S N Bose

The Man and His Work

Collected

Scientific

Papers

S N Bose : The Man and His Work

Part I : Collected Scientific Papers

CENTRE FOR
S. N. BOSE NATIONAL
BASIC SCIENCES 1986
1894-1974
विज्ञानेन परिपश्यन्ति धीरा :

S N Bose (1894-1974), best known for his seminal contribution to quantum statistics, left behind a substantial body of original work in the form of papers in a wide range of fields including chemistry, spectroscopy, thermoluminescence, statistics, group theory, mathematical physics and Einstein's unified theories, collected in the first part of this volume, with annotative introductions on the papers by distinguished scientists from a later generation.

Bose played a significant role in laying the foundations of science teaching and research at the two major universities of eastern India in the first half of the century, Calcutta and Dhaka (now in Bangladesh). It was in Dhaka in 1924 that he devised the new counting method for light-quanta which reconciled Einstein's concept of light-quanta with Planck's law. Within the next six months Einstein wrote three papers showing that Bose's method, when slightly extended to material atoms, yielded the correct quantum theory of ideal gases. The new counting method has come to be known as the Bose-Einstein statistics. Particles whose behaviour is described by the Bose-Einstein statistics have been named bosons after Bose.

As a teacher, Bose developed a fairly comprehensive model for education in India that focussed on the dissemination of a scientific culture in the country through the regional vernaculars; and was equipped to negotiate with the developmental needs of the nation. His lectures and addresses and miscellaneous pieces, collected in the second part of this volume (several of them translated from Bengali for the first time) provide glimpses into his various emotional and intellectual concerns, passions and obsessions, and friendships, documented in the long biography that is the most exhaustive to date, and brings together a lot of material gathered from various sources including the Einstein papers stored at Jerusalem and the archives of Dhaka University, and rarely seen pieces of correspondence.

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S N Bose National Centre for Basic Sciences
Calcutta

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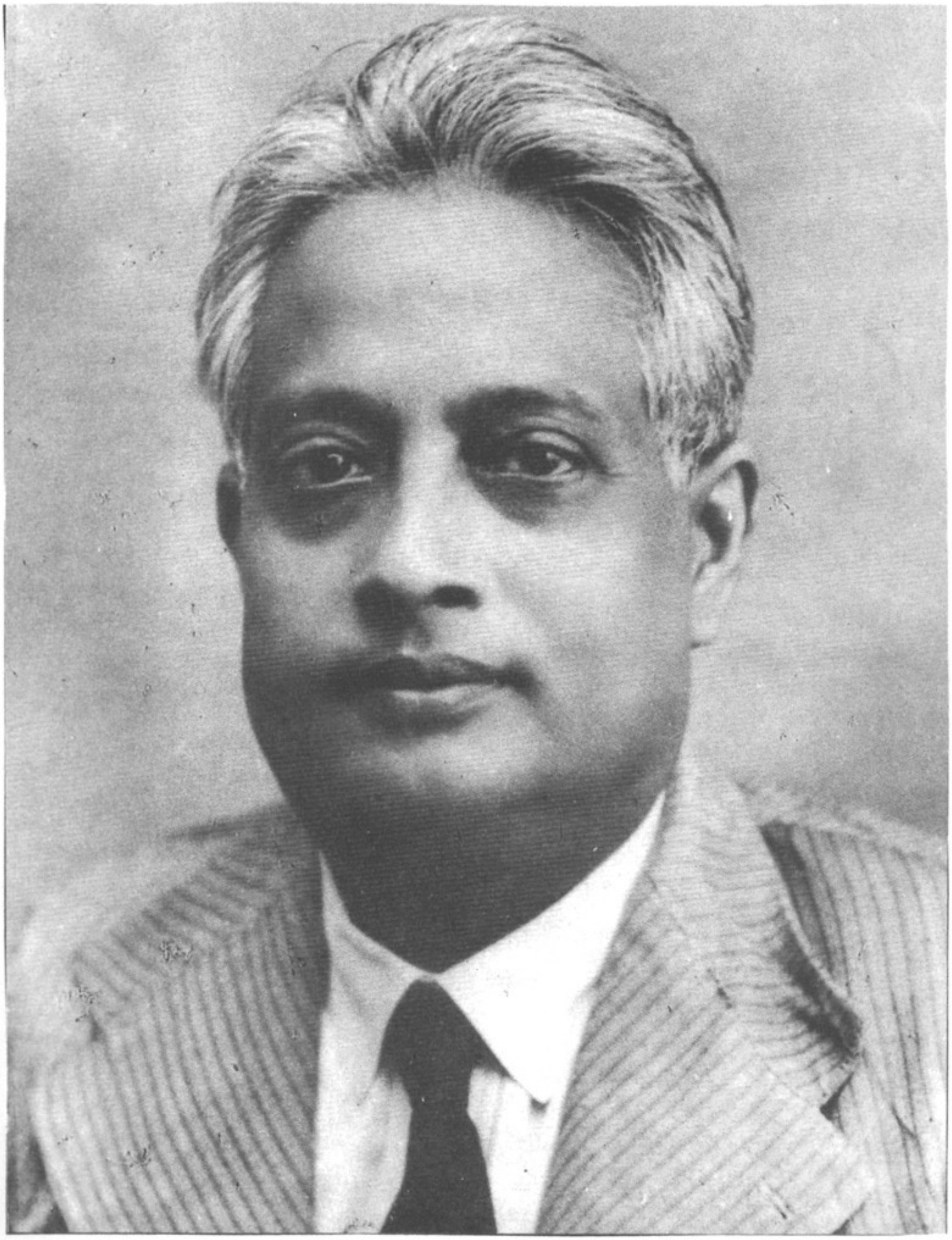

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— for giving us access to information, material and documents in their holdings

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and in particular Rathindranath and Benu Bose, Nilima Mitra, Ruchira Mitra, Ramendranath Bose, and Jayanti Chattopadhyay

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S N Bose : Introductions to scientific papers

Foreword

The Satyendra Nath Bose National Centre for Basic Sciences was established by the Department of Science and Technology, Government of India, in 1986 as a memorial to the Indian scientist Satyendranath Bose. On the occasion of his birth centenary in 1994, the Centre has prepared the present volume incorporating his scientific papers, some selected miscellaneous pieces and addresses, and a biography. Bose wrote and published in four languages : Bengali, English, German and French. Though we have included the original German and French articles, we have provided their English translations also. His writings in Bengali have been collected and brought out by Bangiya Vijnan Parishad, an institution he helped to establish. We have included translations of some of these articles which may be of interest to a general reader.

Several biographical accounts of S N Bose were available. We found that the list of scientific papers was incomplete in all of them, and that not enough attention had been given to his work in Dhaka after his return from Europe. Two published papers — one in German (1927) and another in English (1929) from this period — are included in our collection. Enakshi Chatterjee and Santimay Chatterjee have given a fuller account of his activities in Dhaka — establishing physics laboratories, and helping develop modern research in physics and chemistry.

In some respects S N Bose's career is the story of the triumph and tragedy of Indian science in the first half of the twentieth century. His brief encounter with Europe in the twenties assured his place among the great scientists. He tried to strengthen the scientific base in India but he found little financial and spiritual sustenance. He himself described these two aspects very well, when he said that he was like a comet which came once but never returned, and he seemed to be living on the Moon.

Some comments by later Indian workers about the scientific papers are also included; these hopefully will help an inquisitive young research worker in placing the scientific papers in the right perspective.

A long list of acknowledgements preceding this foreword covers most of the individuals and organizations who have helped us in several ways. Special thanks are due to the Department of Science and Technology, Government of India, for financial support to the S N Bose National Centre for Basic Sciences for activities in the Centenary year, of which the present publication is one.

The volume is divided into two parts — the first part aimed at a more specialist readership, containing his scientific papers, with introductory commentaries by later workers in the specific fields; and the second part designed for a more general readership, with the exhaustive biography, his public lectures and addresses, and miscellaneous pieces (several of them translated from Bengali for the first time).

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1

The Equation of State of a Real Gas

1 *On the Influence of the Finite Volume of Molecules on the Equation of State* (1918)
2 *On the Equation of State* (1920)

In a doctoral dissertation to the University of Leiden in 1873, J D van der Waals proposed a modified equation of state of a real gas. He included two effects : the excluded volume effect due to strong range repulsion and an intrinsic pressure due to weak long range attraction. The work was praised by J C Maxwell who proposed the well-known Maxwell construction to remove the unstable part of the isotherms. The van der Waals equation was enormously successful in explaining critical phenomena, critical indices and the liquefaction of gases. By 1910 experimentalists, however, began to find deviations from the predictions; two constants were not enough, and people started looking for improvements. In 1901 K Onnes introduced the virial coefficients to represent the isotherms. But the simplicity of the approach of an equation of state with a few parameters was too attractive : more than fifty equations of state have been proposed. M N Saha and S N Bose used thermodynamics, especially the Boltzmann formula for entropy, in their characteristic way to arrive at their equation of state. The Boltzmann formula would appear again in Bose's other works. The modern theory of a gas at high density starts from the cluster development of H D Ursell and J E Mayer (J E Mayer and M G Mayer, *Statistical Mechanics*, J Wiley and Sons, New York 1977, Second edition, pp.229 ff.).

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2

Classical Mechanics

1 ***The Stress-Equations of Equilibrium*** **(1919)**
2 ***On the Herpolhode*** **(1919)**

The equilibrium of isotropic elastic solid bodies is discussed in the well-known book by A E H Love, *A Treatise On the Mathematical Theory of Elasticity,* Fourth Edition, Dover, New York 1927, ch V. Here in pp. 134-5, J H Michell's equations are given. V Cerutti's method is described in ch X, pp. 237-40. The problem of the sphere is discussed in ch XI. After the publication of the paper in 1919 Bose lost interest in the problems of elasticity and never returned to them in published work.

The force-free motion of a body about a fixed point is discussed in books of classical dynamics (eg E T Whittaker, *A Treatise on the Dynamics of Particles and Rigid Bodies*, Fourth edition, Cambridge University Press, Cambridge 1961) in terms of elliptic functions. Poinsot's construction gives a simple geometrical description of the motion. Bose proved a geometrical property without using elliptic functions. The same method was also found by W van der Woude.

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3

Quantum Theory

1. *On the Deduction of Rydberg's Law from the Quantum Theory of Spectral Emission* (1920)
2. *Plancks Gesetz und Lichtquantenhypothese* (1924)
3. *Wärmegleichgewicht im Strahlungsfeld bei Anwesenheit von Materie* (1924)
4. *A Note on Dirac Equations and the Zeeman Effect* (1943)

Bose's contribution to Quantum Theory consists of four papers, two of which are on atomic physics and two are about quantum statistical mechanics. This covers the period 1920-1943. Of these, the best known is *the* paper on Bose statistics for the photon gas (*Z f Physik* **26**, 178 (1924)). A brief critical appreciation of these papers follows:

In the first paper *On the Deduction of Rydberg's Law from the Quantum Theory of Spectral Emission*, the Bohr-Sommerfeld method of quantization of action

$$\oint p_j \, dq_j = n_j \, h$$

is used to generate Balmer terms

$$\frac{N}{m + \alpha + (\beta/m^2)}$$

for any series for any atom. Bose starts with a monopole plus a dipole as the best approximation of the potential seen by a valence electron :

$$V(r) = -\frac{e^2}{r} + \frac{e\ L \cos\ \theta}{r^2}.$$

The calculations involve abelian integrals which have to be suitably approximated. Bose also has an effective expression for the Rydberg constant in terms of the fundamental constants, but is either unaware of (or disinclined to cite) A Haas who first obtained the relation (A Haas, quoted in *Introduction to Theoretical Physics*, vol II, Constable, London 1948).

In the paper *A Note on Dirac Equations and the Zeeman Effect* (*Indian J Phys.* **17**, 301 (1943)), written together with K Basu, the problem of the energy levels of a hydrogenic atom in an inhomogeneous magnetic field is solved using Sonine polynomials. The calculations are elegant and straightforward and lead to the quadratic dependence of the perturbed energy levels on the magnetic field.

The celebrated paper *Plancks Gesetz und Lichtquantenhypothese* (*Z f Physik* **26**,168 (1924)) of S N Bose introduces the new photon statistics and deduces the Planck distribution as the equilibrium distribution for photons. Bose considered complexions of photons as the primary characterization of the state rather than which photon had which energy. Photons were treated as strictly indistinguishable.

With the modification that Bose introduced into the calculation we obtain the correct thermodynamics of the photon gas. That one step was the basis of the new synthesis between the wave and the particle properties of photons, and with it, the foundations of quantum field theory. After Bose's paper came an avalanche of developments: the extension of Bose's theory to particles of arbitrary mass and non-zero chemical potential by Einstein, the Fermi-Dirac statistics for electrons, the quantization of the electromagnetic field by Heisenberg and Pauli, and quantum electrodynamics by Dirac.

To put Bose's synthesis in its proper setting it is good to recall that in the nineteen-twenties there were two items of unfinished business, one regarding photons as particles and the other concerning statistical mechanics of identical particles. In both cases uneasy makeshift solutions were generally accepted instead of definitive solutions. In the old dichotomy between particles and waves for describing light there were highly persuasive arguments on both sides, but it was thought that a crucial experiment was the determination of the relative speed of light in two media with different refractive indices. It appeared that if light consisted of particles, the speed in the optically denser medium should be greater; if it consisted of waves,this speed should be lesser. Fizeau's experiment on the speed of light in water thus seemed to find evidence definitively for the wave theory! Yet, what of the postulated photons? How do we reconcile the notion of a photon with its discrete momentum and energy to its lesser speed in water than in air? We must conclude that photons may be particles, but they do not behave as particles are naïvely expected to behave. A revision of the concept of a particle ought to be made. We now know that extended particles also bend towards the normal in a medium in which it moves slower; so Fizeau's experiment could not really distinguish between extended particles and waves.

The other item of unfinished business is even older. It concerns itself with the statistical mechanics of identical particles. In calculating the partition function and the entropy, one finds that the entropy is not a strictly additive quantity: when we mix two volumes of an ideal gas at the same temperature and pressure the resultant entropy is larger than the sum of the two entropies.This Gibbs' paradox shows that such a collection of identical particles is not a satisfactory model for an ideal gas. Instead of heeding this warning signal people 'fixed' the trouble by an ad hoc procedure in dividing the partition function by the factorial of the number of particles, thus condoning the Maxwell--Boltzmann statistics. The genuine need for a reexamination of the implications of the strict identity of the particles was not appreciated until Bose, three decades yet to be born.

Elementary particles were originally introduced as the stuff from which the world was made. They were to be immutable entities. But the photon was clearly an entity which could be created or destroyed. Where does a photon come from and where does it go? And how can we really understand creation and destruction? What is the implication of strict identity of photons? In what sense and to what extent can we think of light as a collection of photons? All these questions were answered at one stroke by Bose, who asked us to consider the many-photon states to be counted as states with equal probability. Photons were thus particles all right, but particles for which strict identity was to be recognized by considering as distinct only those cases in which the distribution of photons over phase cells were distinct.

Photons thus became nothing but levels of an underlying field. Creation or destruction of photons is merely a 'movement' of the field. Photons are then manifestations of the potentialities of the radiation field: the dichotomy between the field and the particle thus ceases. Two have become one.

Automatically the embarrassment of the Gibbs paradox is resolved: the paradox was just telling us that the strict identity of particles must be taken into account. But if photons are but the differences between the levels of the radiation field, they are all identical! And the process of creation and destruction is thought of as a change in the state of the field, the 'motion' of the field. If we have equations of motion of the field, we have the means of describing the creation and destruction of photons. It took two more years for Heisenberg and Pauli to write down the equations of motion for the radiation field and another year for Dirac to construct a theory of the emission and absorption of photons. In Dirac's work the oscillators of Planck were at last identified. The formulation of the equations of motion of the electromagnetic field had still unsatisfactory features. Many others contributed to the resolution of this problem, among them Dirac, Heisenberg and Pauli, Fermi and Gupta.

In the course of his work on the quantum theory of radiation Dirac introduced the now familiar notion of creation and destruction operators which increase or decrease the number of quanta in a state. These creation and destruction operators, introduced as the operator coefficients of the quantized field operator, do not commute with each other but instead satisfy a commutation relation which transcribes the commutation relations between field quantities as formulated by Heisenberg and Pauli. Dirac had already discovered that the commutator bracket in quantum mechanics was the natural analogue to the Poisson bracket in classical mechanics. It reaffirms the Bose hypothesis that photons obey the Bose statistics.

When Bose advanced his hypothesis the only species of elementary particles that were identified were the electron, the proton, the neutron and the photon. Of these only the photon obeys Bose statistics. The photon number is not covered and it is a zero mass particle. Both of these impart special characteristics to the statistics of photons. Among the Bose systems available then was Helium. In this case the particle number is conserved; and the particles are nonrelativistic. So we need to extend Bose's ansatz. This was done by Einstein in the same year. To conserve the particle number we have to introduce a non-zero chemical potential. For photons the chemical potential is zero. In the case of an ideal Bose gas with non-zero chemical potential there is a critical temperature below which a finite fraction of the gas condenses into a single quantum state. This condensed phase, discovered by London, should exhibit superfluid properties; and London suggested that superfluid helium should be related to this Bose-Einstein condensation phenomenon.

Einstein arranged for the translation and publication of Bose's paper on the statistics of photons and added a remark endorsing it as 'substantial progress.' Both in his original letter to Einstein and in his subsequent correspondence Bose addressed the great man as 'teacher' and accords him great respect; and that is as it ought to be. It is in the definition of the teacher, as understood in the classical Indian tradition, that he remove all the doubts of the student and weld his understanding into a harmonious unity: such a teacher is the one worthy of adoration.

> To that Teacher who removes all my doubts, welds my vision into a unity
> and thus enables me to gaze on secret knowledge; to that One my homage.

Einstein does not seem to have told Bose how his theory could be extended to a theory of ideal Bose gases by introducing a chemical potential and making use of a general energy–momentum relation. Einstein formulated this extension in one of his papers.

Bose followed up this paper by another more detailed and more ambitious paper. In his first paper, Bose had used a 'static derivation' of an equilibrium configuration as the most probable configuration. In the second paper, he used a 'dynamic derivation' in which the equilibrium configuration is the one in which transitions into and out of each state balance each other. For the special model of a two-level Bohr atom and monochromatic radiation, Einstein had shown (*Phys Z* **18**, 12 (1917)) how one can understand the Planck distribution when one takes into account both the stimulated and the spontaneous emission on the one hand and stimulated absorption on the other.

Bose generalized this to arbitrary atoms with arbitrary numbers of discrete energy levels and radiation of all possible frequencies: the essential elements in the derivation are the conservation of energy in collisions and the ratio of the rates of emissions and absorption. These were correctly computed by Bose. (Unfortunately, for some strange reason, Bose seems to have different values for the absolute transition rates which prompted Einstein to append a critical comment to the paper.)

In these two papers done before the birth of quantum mechanics as we know today, Bose laid the foundations of a quantum theory of the electromagnetic field.

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4

Chemistry

1 *Messungen der Zersetzungsspannung in nichtwässerigen Lösungsmitteln* (1927)
2 *Reaction of Sulphonazides with Pyridine* (1943)

To an outsider it might seem odd that a theoretical physicist of Bose's stature, who was intensely trying to understand the basic laws of physics, would now and then take time off and spend days in a dingy chemical laboratory. One might reasonably expect a theoretical physicist like him to get interested in the nature of the chemical bond or in the properties of electrolytic solutions — a problem fruitfully tackled by his friend and classmate J C Ghosh; but strangely, he showed no interest in any of these theoretical challenges. Rather, he liked to synthesize and analyze chemicals useful to contemporary society. He was obviously guided by a nationalistic feeling. A part of his interest in down-to-earth chemistry might have been inherited from his father who founded a small chemical industry. His close association in his formative days with Acharya P C Ray, who championed the cause of Indian chemical industries, might also have played a role.

In Dhaka University he set up a working organic chemistry laboratory and encouraged his students to synthesize a number of important drugs like emetine, sulpha drugs, etc.Most of these works are either not published at all or published without his name. Only occasionally did his name appear, such as in *Science and Culture* with P K Dutta, where the reaction between sulphonazides and pyridine was studied. His interest in organic chemistry continued after his return to Calcutta University. In an article in *Science and Culture* **10**, 1974, p. 295, A Chatterjee has recalled some of Bose's forays in organic chemistry, particularly alkaloid chemistry.

At Dhaka, Biswas and Bose developed a simple instrument to measure the polarization voltage and decomposition voltage of some electrolytes in non-aqueous solvents, where a simple manual commutator was used to alternate the direction of current and thus reduce polarization errors at the electrodes. At Calcutta University he set up an X-ray laboratory and helped chemists with structural problems. Professor J N Mukherjee, the noted colloid chemist and classmate of Bose, prepared a map of clays of different parts of India. The structural aspects of these clays were studied in Bose's laboratory. He encouraged the organic chemists to determine molecular structures through X-ray analysis.

Although Bose's works in chemistry do not carry the mark of a genius, these show the versatility of the man and his eagerness to solve the problems of the country as well as his colleagues and students.

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3 *Germanium in Sphalerite from Nepal* (1950)
4 *Extraction of Germanium from Sphalerite Collected from Nepal—Part I* (1950)
5 *Extraction of Germanium from Sphalerite Collected from Nepal—Part II* (1950)

The invention of the transistor brought to reality the inventor's dream of realizing an electron device by controlling the motion of electrons available in ample quantity inside a solid. Power was no longer required to be wasted in obtaining controllable electrons in vacuum. The active device for electronics could now be made much smaller and the life expectancy of electronic equipment made much larger.

The first transistor was made with polycrystals of germanium (element number 32 in the Periodic Table) discovered by the German chemist Winkler in 1886 and named after his country. As germanium transistors were expected to replace vacuum tubes in all electronic circuits, S K Mitra concluded in his Presidential Address at the Forty-second Indian Science Congress held in 1955 at Baroda that 'the future stage, if one may venture to make a prophecy, will be the era of the uses and applications of the element germanium.' But germanium was mostly recovered from chimney dust collected from refineries of lead, titanium and zinc in the USA and in the UK from chimneys of all kinds of industries using coal which in Britain had a large germanium content. There were not many other sources of germanium and so scientists were exploring all possible sources. It is no wonder that Bose, being aware of all current developments, endowed with a very sharp foresight, and committed to the national interests, encouraged his student R K Datta to look for sources of germanium in minerals. This work resulted in three letters, in which is described the procedure used for extracting germanium from sphalerite collected from Nepal and the suitability of the ores as a source of germanium. The three letters should be considered as important original contributions to the technology of germanium and establishing sphalerite as a good source of germanium. The only pity is that the work was not apparently followed up, and no semiconductor industry grew in India, although Bose did identify the problem within a year of the invention of the transistor.

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5

Spectroscopy

Beryllium Spectrum in the Region λ 3367-1964 (1929)

The spectrum of the beryllium atom is similar to that of the helium atom and should have been easier to analyze. It is well-known that He I shows singlets and triplets and He II (with one electron detached) shows doublets. But early observations did not resolve the triplets clearly. One sees the same kind of controversy in Be I and Be II in this paper.

The way the spectra were clarified by earlier workers seems almost miraculous today. We also note that after returning from Europe in 1926, Bose organized a spectroscopic laboratory at Dhaka. (Another paper published in 1927 deals with experimental work in electrochemistry.)

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6

Statistics

1 *On the Complete Moment-coefficients of the D^2-statistic* (1936)
2 *On the Moment-coefficients of the D^2-statistic and Certain Integral and Differential Equations Connected with the Multivariate Normal Population* (1937)

Though Statistics — as a scientific method — was known in India even at the beginning of the present century, significant statistical studies and researches were initiated during the twenties by the late Professor P C Mahalanobis. While working as a Professor of Physics at Presidency College, Calcutta, Mahalanobis took up several data-analytic studies on a wide variety of real-life problems and followed them up to develop several theoretical models and inferential tools. He could inspire quite a few brilliant young men in physics and mathematics to work on theoretical as well as applicational aspects of Statistics. Some of those drawn to Statistics — directly or indirectly by Professor Mahalanobis — later changed their initial academic pursuits to concentrate on Statistics. Others, like Professor Satyendranath Bose, continued with their original vistas but made remarkable contributions to statistical theory and absorbed statistical concepts and tools in their subsequent works.

While working on some anthropometric data on statures of Anglo-Indians, Professor Mahalanobis (1936) came up with an idea of generalized distance between two populations — more or less on the lines of Karl Pearson's coefficient of racial likeness (1921). He defined the famous D^2-statistic for this purpose in terms of the mean vectors (of the characteristics measured in the two groups) and the covariance matrix. It was intended to be a quantity determined entirely in terms of the sample values of the variates. And for judging the significance or otherwise of such a sample value of the statistic its sampling distribution had to be worked out. R C Bose (1936) derived an exact distribution of a modified form of the D^2-statistic in which population variances and covariances were substituted for the corresponding sample estimates. He used the expression of the density function (in terms of Bessel functions) to obtain the moment coefficient, which turned out to be the same as those calculated earlier by Professor Mahalanobis (1936), using approximate methods.

Fascinated by the new idea of generalized distance and the nuances of the mathematics used by R C Bose, S N Bose (then working in the University of Dacca) looked at the problem (1936) and came up with a recurrence formula for moments of the modified D^2-statistic to obtain the exact moments without using the density expression and investigated some properties of the moment coefficients. He could also offer a different equation which can yield moments of even fractional orders (expressible not by a polynomial but by an infinite series) — entities that have found many important uses later.

Professor Bose continued his interest in the mathematics of this problem. In his 1937 *Sankhyā* paper he started with some algebraic identities among the moments of the modified D^2-statistic deduced from their differential forms and derived an integral equation connected with these moments. He investigated the fundamental differential equation satisfied by the multivariate normal distribution in its various forms — spherical, parabolic and product. He also offered a series solution of the differential equation.

The contents of the two papers in Sankhyā (1936 and 1937) speak of the depth of S N Bose's vision about a new problem and its mathematical fallouts. It may be pointed out, however, that the results derived by S N Bose have not created a big impact on statistical aspects of multivariate analysis, since the basic shortcoming of the modified D^2-statistic considered by him renders the results somewhat less useful. The differential equation satisfied by the multivariate normal distribution has not found its way into later investigations on characterizations. In fine, the papers by S N Bose on D^2-statistic have been rarely cited or used.

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3 Rajchandra Bose : 'On the Exact Distribution and Moment- coefficients of the D^2-statistic', *Sankhyā* **2**, Part 2 (1936). 143-154.
4 Rajchandra Bose : 'On the Distribution of Differences of Mean Values in Two Samples and the Definition of the D^2-Statistic', *Sankhyā* **2**, Part 4 (1936), 379-384.

5 Satyendranath Bose : 'On the Complete Moment-coefficients of the D^2-statistic, *Sankhyā* **2**, Part 4 (1936), 385-396.

6 Satyendranath Bose : 'On the Moment-coefficients of the D^2-statistic and Certain Integral and Differential Equations Connected with the Multivariate Normal Population', *Sankhyā* **3**, Part 2 (1937), 105-124.

7

The Ionosphere

1 *Anomalous Dielectric Constant of Artificial Ionosphere* (1937)
2 *On the Total Reflection of Electromagnetic Waves in the Ionosphere* (1938)

Various theories of the propagation of radio waves through the ionosphere were developed during 1930-1938 based on either the ray treatment or the wave treatment. The refractive index comes out in general to be a complex quantity which is a function of the electron number density and collision frequency which are functions of height. Consequently, the solution of Maxwell's equations becomes highly complex. Appleton deduced certain conditions of reflection by assuming that the refractive index must vanish.

However, since the refractive index is complex, other criteria were formulated by various workers (D R Hartree, *Proc Camb Phil Soc* **25**, 47,1929: ibid. **27** 143, 1931; *Proc Roy Soc* A **131**, 428, 1931; Fosterling and Lassen, *Ann d Physik* **18**, 26, 1933; H G Booker, *Proc Roy Soc* A **155**, 235, 1936; M N Saha and R N Rai, *Proc Nat Inst Sci Ind* **3**, 359, 1937; M N Saha, R N Rai and K B Mathur, *Proc Nat Inst Sci Ind* **4**, 53, 1938).

R N Rai (*Proc Nat Inst Sci Ind* **3**, 307, 1937) suggested that a better criterion of reflection would be that the group velocity of the wave vanishes. This gave him a new condition in addition to those of Appleton, a condition that was experimentally confirmed by Pant and Bajpai (*Science and Culture* **2**, 409, 1937), L Harang (*Terr Mag* **40**, 29, 1937) and R Jonaust, Abadie and Joigny (*L'onde electrique* **16**, 185, 1937).

Experiments also indicated that apart from totally reflecting electromagnetic waves the ionosphere also partially reflected and partially transmitted these waves. Moreover, for very long waves, the complex refractive index can change within one wavelength and reflection can occur even though the refractive index is not zero. These features could not be explained by any of the theories.

Both Appleton's and Rai's conditions of reflection were obtained by neglecting the effect of damping, an essential feature of the physical process. The questions that arose were: (1) do both these conditions follow as consequences of the Maxwell-Lorentz theory and (2) what are the conditions of reflection when collisional damping cannot be neglected? These are the problems that Bose set about to tackle in the 1938 paper.

Instead of Maxwell's equations with a complex refractive index, Bose used the microscopic equations of the Maxwell-Lorentz theory and the method of characteristics used by Hadamard, Debye and others. The method is general and can also be used for the propagation of light in a material medium. However, Bose's results were expressed in symbols unfamiliar to workers in ionospheric physics. It was M N Saha and K B Mathur (*Ind Jour Phys* **13**, 251, 1939) who gave a critical assessment of Bose's results in a form easily comprehensible to them. They showed that Bose's treatment gives the same results as those of previous workers for the case of vertical propagation. However, in the presence of collisions the nature of the complex refractive index and polarization became complicated, and Bose's conditions do not give accurate results.

In 1960 H K Sen and A A Wyller (*J Geophys Rev* **65**, 3931, 1960) used the Chapman-Enskog method and the Boltzmann equation to obtain the effect of the velocity dependence of the collision frequency in a closed analytic form.

Earlier in 1937 Bose and S R Khastgir had investigated the conditions under which the value of the dielectric constant of an ionized medium exceeds unity (anomalous behaviour) and carried out simple experiments with ionized air in a discharge tube (artificial ionosphere) to show that the dispersion formula alone cannot explain the observed anomalies.

EDITORS

8

Mathematical Physics

1 *Studies in Lorentz Group* (1939)
2 *The Complete Solution of the Equation*:
$\nabla^2\varphi - \frac{\partial^2\varphi}{c^2\partial t^2} - k^2\varphi = -4\pi\rho(xyzt)$ (1941)
3 *On an Integral Equation Associated with the Equation for Hydrogen Atom* (1945)

The three papers are on various problems in mathematical physics: the first deals with the properties of the matrix group O (4,C) in its defining representation, with consequences for the physical Lorentz group $\mathrm{SO}^{\uparrow}$(3,1); the second develops a particular technique, involving integration in the complex plane à la Sommerfeld, for solving the inhomogeneous Klein-Gordan equation; and the third is concerned with the Schrödinger equation for the Coulomb problem, analyzed as an integral equation in momentum space. These papers were written in 1939, 1941 and 1945 respectively. A few introductory comments regarding each are presented.

It is most easily seen from the infinitesimal approach, provided that one works with complex linear combinations of the basis elements, that the Lie algebra of SO(3,1) formally splits into two commuting SU(2) or angular momentum like algebras. This helps in the construction of all the finite dimensional irreducible matrix representations of SO(3,1), and the connection of SL(2,C) also becomes clear. One sees on the way the possibility of formally expressing a finite element as a product of two commuting factors, in the complex sense. To realize all this without using infinitesimal techniques, however, requires considerable ingenuity. This is what is attempted in the 1939 paper of Bose. He in fact examines the much larger group of all complex orthogonal transformations in four dimensions, namely the group O(4,C). The key idea is to deal with symmetric elements of O (4,C) and find ways of characterizing and factoring them; and for antisymmetric matrices, exploiting the properties of self-dual and antiself-dual sets of matrices. It is the latter that actually lead to the commuting structure referred to above. From the discussion of O(4,C), Bose descends to the case of SO(3,1) by imposing the requisite reality conditions, and thus connects up the spinor approach based on SL(2,C), as typified in the familiar statement (1/2,1/2) = (1/2,0) x (0,1/2) : the four-vector representation of the Lorentz group is the product of the two basic spinor representations.

Turning to the second paper, it is basically an exercise using relativistically invariant solutions of the Klein-Gordan equation to solve a problem for which H J Bhabha in an earlier paper had presented a solution in a particular form. At the time the paper was written, namely 1941, the technology associated with the wave equation using Stückelberg-Feynman functions, invariant Green's functions, retarded and advanced ones, etc., were presumably not yet widely known and used. The interest in Bose's paper lies in its use of complex contour integration methods, a favourite of Sommerfeld, to get solutions of the inhomogeneous Klein-Gordan equation in Kirchhoff-like form.

The third (1945) paper is devoted to a discussion of the quantum mechanical Coulomb problem in momentum space. It has been known for a very long time, through the work of V Fock and V Bargmann, that if the Schrödinger equation for energy eigenfunctions is expressed in momentum space, it has the form of an integral equation, and moreover it makes the higher symmetry in the problem manifest. Thus the SO(4) symmetry for bound states, and SO(3,1) for scattering states, can be explicitly exhibited by suitable energy-dependent choices of variables. (It is curious, however, that Bose does not refer to Fock's and Bargmann's papers at all). Bose however proceeds somewhat differently realizing that the energy-dependent choice of variables would discriminate between bound and scattering states. He exploits methods of Hobson involving solid harmonics and operator calculus to explicitly solve the integral equation, after reducing it essentially to a radial problem (in momentum space). An interesting feature of his expressions is that the quantization of the bound state energies, and also the fact that the angular momentum quantum number is bounded above essentially by the principal quantum number, both arise from the requirement that the momentum space eigenfunction be single-valued, rather than that it be normalizable. It is the latter condition that is normally used in a configuration space solution to the problem. His method of course leads to both bound and scattering state wave functions. At the end of the paper, a completeness statement involving the former alone is developed. One presumes therefore that this is distinct from the usual physical notion of completeness of eigenfunctions of the Hamiltonian operator, since that would have brought in the scattering states also.

All three papers show a taste and knack for clever algebraic and analytic manipulations, even though the problems addressed are circumscribed in scope.

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9

Unified Field Theory

1 *Les identités de divergence dans la nouvelle théorie unitaire* (1953) [30.3.53]
2 *Une théorie du champ unitaire avec* $\Gamma_\mu \neq 0$ (1953) [18.7.53]
3 *Certaines consequences l'existence du tenseur g dans le champ affine relativiste* (1953) [18.7.53]
4 *The affine connection in Einstein's new unitary field theory* (1954) [29.9.52]
5 *Solution d'une équation tensorielle intervenant dans la théorie du champ unitaire* (1955) [Receipt date not mentioned]

The papers are listed in chronological order. The journal receipt date is given within brackets.

The general theory of relativity established itself as a satisfactory description of gravitation. The only other interaction that was then known was the electromagnetic, and a natural sentiment was to bring the electromagnetic and gravitational interactions under one umbrella — to be more precise, to geometrize the electromagnetic field as well.

Einstein, however, demanded much more from a unified field theory. He was committed to the field idea and was never reconciled to quantum mechanics with its basic indeterminacy. He expected that matter itself would ultimately resolve into a field and, as singularities mean a breakdown of the field concept, there should be no singularities in a truly unified field theory. Again, when Kaluza's five-dimensional geometry appeared, the idea was that our observations were limited to four dimensions and thus were essentially of an incomplete nature. One might wonder whether this incompleteness could account for quantum indeterminacy.

What, by the way, is a unified theory ? We may quote Einstein's definition :

> Neither the field equation nor the Hamiltonian can be expressed as a sum of several invariant parts but are formally unified entities.

There were a host of unified field theories — one may compare the ever-changing theories with the changing models of cars. But they all had one similarity. The 4-dimensional Riemannian geometry has ten independent variables (the metric tensor components g_{ik}), just sufficient for the gravitational field whose source, the energy--momentum--stress tensor, has also ten components. So, to incorporate the electromagnetic field, one must have some additional geometric variables. These additional variables were introduced in a number of ways, e.g.

(a) In the Weyl theory, it was postulated that the norm of a vector changes in a parallel displacement according to the law.

$$\frac{\delta l}{l} = \phi_i \, dx^i$$

and the vector ϕ_i was identified with the electromagnetic potential vector.

(b) In Kaluza's five-dimensional formalism the metric tensor has 15 components.

(c) The fundamental tensor g_{ik} was assumed to be complex, so that there were 20 variables.

(d) The fundamental tensor was assumed to be non-symmetric and the affinities Γ^{i}_{kl} which define parallel displacement were also taken to be non-symmetric. There were thus altogether 80 variables.

Over a number of years different forms of the non-symmetric theory were developed by Schrödinger and Einstein. One particular form gained wide publicity as it was announced by Einstein first in the popular press in December 1949 and then presented in the 1950 edition of his book *Meaning of Relativity* as an Appendix. In his papers Bose used the adjective 'new' before unified field theory without indicating specifically which particular theory he had in mind. It seems clear that he referred to this theory.

We give a table to pinpoint the salient points of this theory as compared to the general theory of relativity :

	GTR	UFT
1.	Symmetric metric tensor $g_{\mu\nu}$ and symmetric affinities $\Gamma^{\alpha}_{\mu\nu}$.	Non-symmetric $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$. [What is the metric tensor ? No specific statement is made in the book but it became common to identify the symmetric part of $g_{\mu\nu}$ with the metric tensor and the antisymmetric part was related to the electromagnetic field, as we shall presently see.]

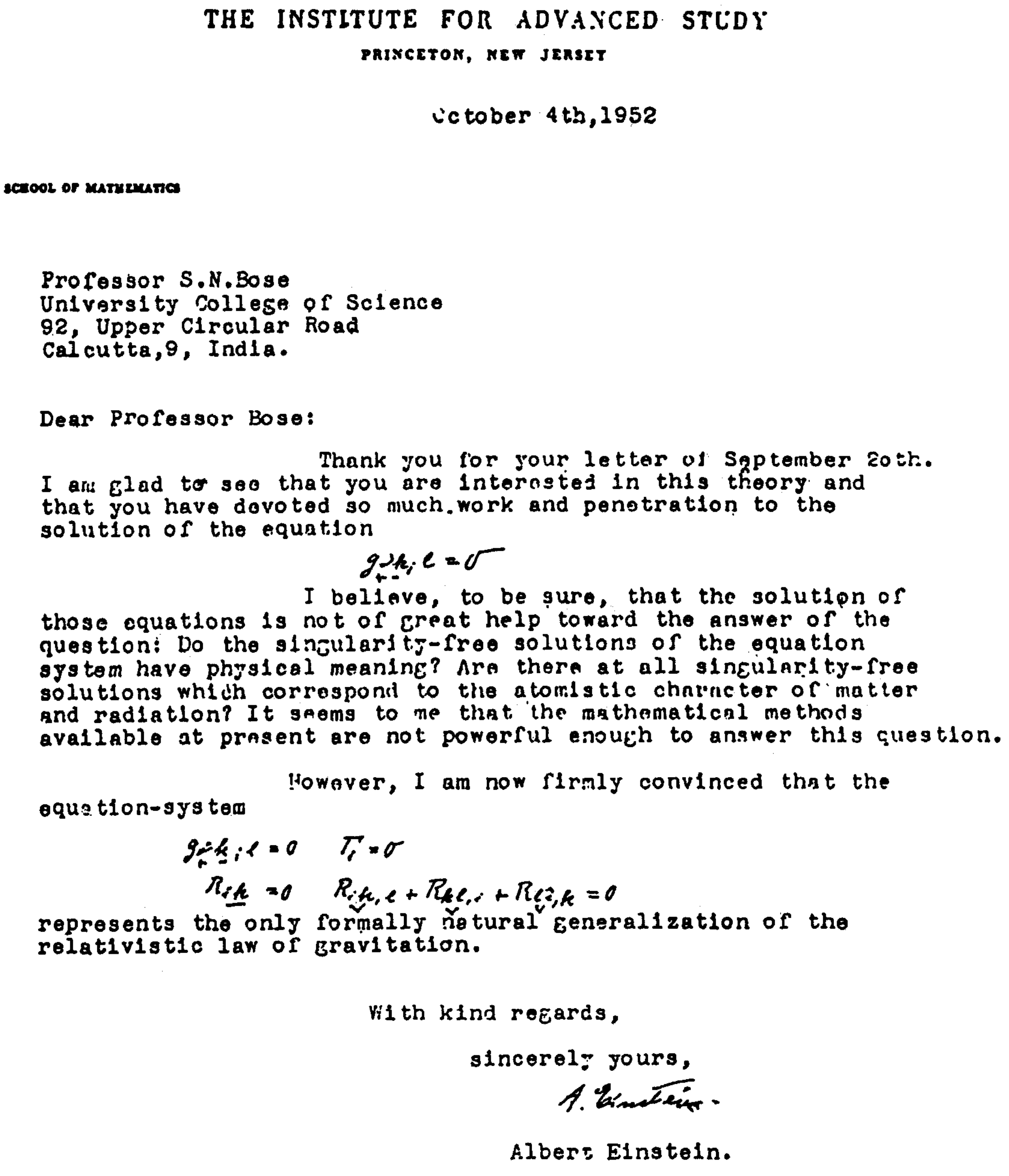
2.	$g_{\mu\nu;\alpha} = 0 \rightarrow$ $\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,\sigma}+g_{\alpha\sigma,\nu}-g_{\nu\sigma,\alpha})$	$g_{\mu\nu;\alpha} = 0$ (*i.e.* $g_{\mu\nu,\alpha}-g_{\rho\nu}\Gamma^{\rho}_{\mu\alpha}-g_{\mu\rho}\Gamma^{\rho}_{\alpha\nu} = 0$). [The solution of these 64 equations, i.e. expressing Γ's in terms of $g_{\mu\nu}$ and their derivatives, was an extremely difficult problem and was given independently by Bose, Einstein and Kaufman, Hlavaty and Tonnelat. Later there were also other workers in the field including R S Mishra.]
3.	Field equations obtained from the variational principle $\delta\int g^{\mu\nu} R_{\mu\nu} \sqrt{-g}\ d^4x = 0$ where $R_{\mu\nu}$ (the Ricci tensor) is $R_{\mu\nu} = \Gamma^{\sigma}_{\mu\alpha}\Gamma^{\alpha}_{\sigma\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} = R_{\nu\mu}$. The variation can be performed in two ways : (1) $g^{\mu\nu}$'s are to be varied and the relation between Γ's and $g_{\mu\nu}$'s assumed. (2) $g^{\mu\nu}$ & Γ's are independently varied—the relations between Γ's and $g_{\mu\nu}$'s appear as one set of field equations.	Field equations obtained from the variational principle $\delta\int g^{ik} u_{ik} \sqrt{-g}\ d^4x = 0$, $g^{\mu\nu}$'s & Γ's varied independently. The tensor u_{ik} is $u_{ik} = R_{ik} - \frac{1}{3}\left[\left(\Gamma_{i,k} - \Gamma_{k,i}\right) + \Gamma_i \Gamma_k\right]$ $\Gamma_i = \Gamma^{l}_{\underset{\vee}{il}}$ and $R_{ik} = \Gamma^{s}_{il}\ \Gamma^{l}_{sk} + \frac{1}{2}\left[\Gamma^{s}_{\underline{is},k} + \Gamma^{s}_{\underline{ks},i}\right] - \Gamma^{s}_{ik}\Gamma^{t}_{\underline{st}} - \Gamma^{s}_{ik,s}$ where underlining indicates the symmetric part and the hook the antisymmetric part. The tensor u_{ik} is selected by some ad hoc conditions introduced by Einstein which may be summarized in the form $u_{ki} = u_{ik}(\tilde{\Gamma})$ $u_{ik} = R_{ik}(\Delta)$ where $\Delta^{l}_{ik} = \Gamma^{l}_{ik} - \frac{1}{3}(\Gamma_i \delta^{l}_{k} - \Gamma_k \delta^{l}_{i})$ so that $\Delta^{l}_{\underset{\vee}{il}} = 0$. The conditions were not justified by any theoretical consideration. *Bose therefore sought to work with a different variational principle.* However, this led to a much more complicated set of equations which involved two arbitrary constants.

4.	Field equations $R_{\mu\nu} = 0\,.$ The equations satisfy four differential identities.	Field equations $R_{ik} = 0\,,$ (a) $g_{\underset{+}{i}\underset{-}{k};l} = 0\,,$ (b) $\Gamma_i = 0\,.$ (c) The equation $\Gamma_i = 0$ was introduced ad-hoc on the ground that this does not cause an overdetermination. The equations (b) and (c) together give $(\sqrt{-g}\, g^{\underset{\vee}{ik}})_{,k} = 0$. This led Einstein to identify $g^{\underset{\vee}{ik}}$ with the dual of the electromagnetic field tensor—there is no free magnetic pole.
5.	If fields other than gravitational were present, additional terms were introduced in the Lagrangian and the field equations then became $R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = k T_{\mu\nu}.$	The electromagnetic field is already taken care of. No mention is made of other fields. Unification is thus not complete.
6.	Physical conclusions drawn all agree with observations.	No physical conclusions could be drawn. Indeed, Pauli remarked (1958), 'Whether the field equation. . . can actually be connected with physics at all is rather doubtful.'

Before going over to the subject matter of these papers, it is somewhat interesting to note that at least four of these papers were produced within the span of less than a year. That was something unusual in Bose's career. Never before had he written so many papers in a year and never before had he contributed as many as five papers in a single field. One may wonder whether these works kindled Bose's enthusiasm to a degree he had never experienced in his whole life. And mark, five papers in four different journals — was it all fun with Bose !

The first paper concerned the divergence identities in the unified field theory we have just outlined. Long before, Hilbert (1924) had proved that the set of equations obtained from a variation principle would not be independent but there would be four divergence identities satisfied by them. For non-symmetric theories Einstein obtained the identities by a method which was rather complicated. Bose showed that they could be obtained in a somewhat more general manner by following Hilbert's procedure.

In the second paper Bose used a much more complicated Lagrangian. Einstein's Lagrangian was obtained on the basis of somewhat arbitrary arguments and then the equation $\Gamma_i = 0$ appended, justifying this by the rather curious reason that it would not make the equation set overdetermined. Bose's equations were naturally more complicated and were not supplemented by the ad hoc condition $\Gamma_i = 0$. However, the equation system involved two arbitrary constants and hence they were not quite appealing. Neither Bose nor any of his students investigated these equations.



THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY

October 4th,1952

SCHOOL OF MATHEMATICS

Professor S.N.Bose
University College of Science
92, Upper Circular Road
Calcutta,9, India.

Dear Professor Bose:

Thank you for your letter of September 20th. I am glad to see that you are interested in this theory and that you have devoted so much work and penetration to the solution of the equation

$$g_{\underset{+}{i}\underset{-}{k};l}=0$$

I believe, to be sure, that the solution of those equations is not of great help toward the answer of the question: Do the singularity-free solutions of the equation system have physical meaning? Are there at all singularity-free solutions which correspond to the atomistic character of matter and radiation? It seems to me that the mathematical methods available at present are not powerful enough to answer this question.

However, I am now firmly convinced that the equation-system

$$g_{\underset{+}{i}\underset{-}{k};l}=0 \qquad \Gamma_i=0$$

$$R_{\underline{ik}}=0 \qquad R_{\underset{\vee}{ik},l}+R_{\underset{\vee}{kl},i}+R_{\underset{\vee}{li},k}=0$$

represents the only formally natural generalization of the relativistic law of gravitation.

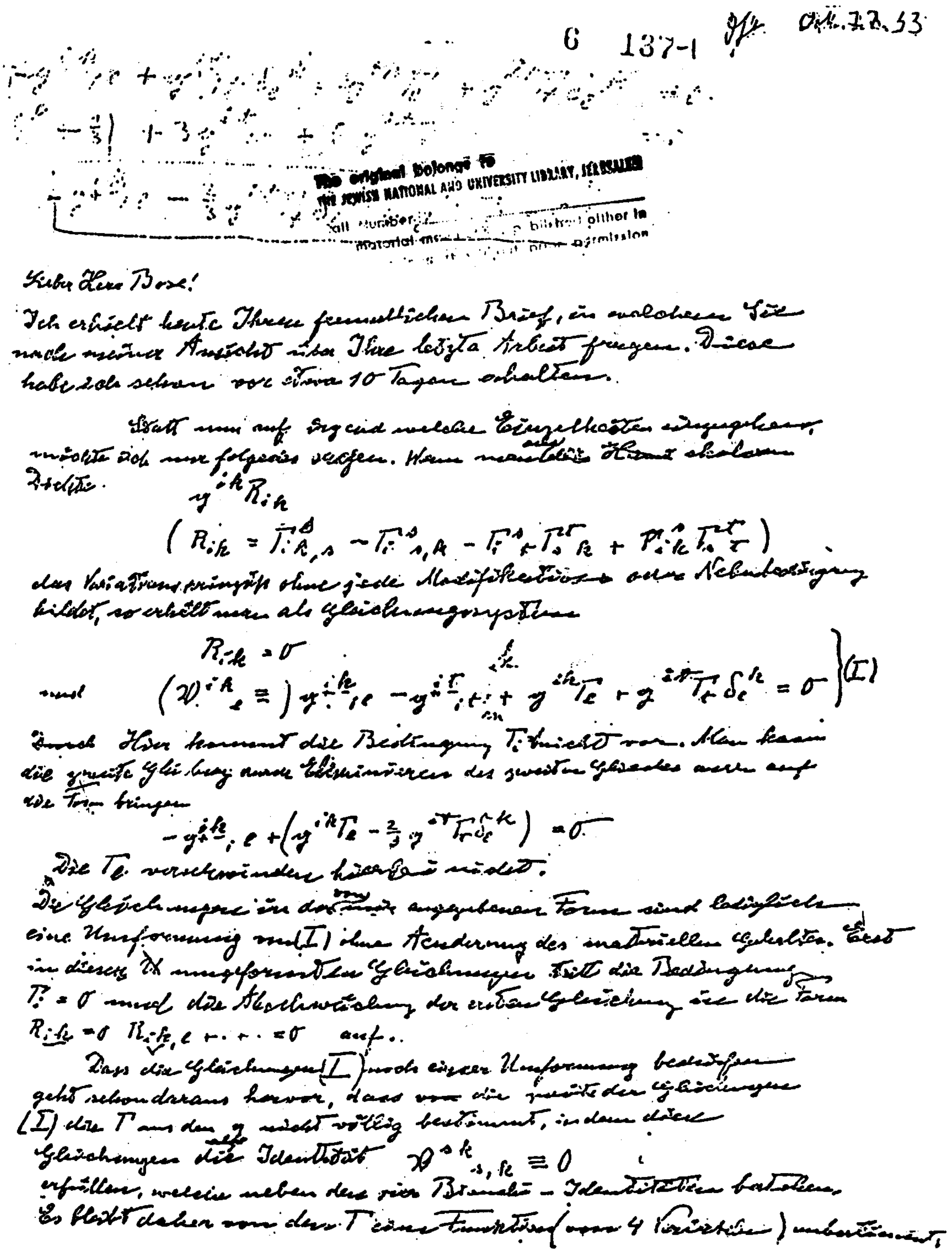
With kind regards,

sincerely yours,

A. Einstein.

Albert Einstein.

Einstein's letter of 4 October 1952 to Bose



Oct. 22.53

Lieber Herr Bose!

Ich erhielt heute Ihren freundlichen Brief, in welchem Sie nach meiner Ansicht über Ihre letzte Arbeit fragen. Diese habe ich schon vor etwa 10 Tagen erhalten.

Statt nun auf irgend welche Einzelheiten einzugehen, möchte ich nur folgendes sagen. Wenn man aus der ~~Hamilt~~ skalaren Dichte

$$\mathfrak{g}^{ik}R_{ik}$$

$$\left(R_{ik} = \Gamma^{s}_{ik,s} - \Gamma^{s}_{is,k} - \Gamma^{s}_{it}\Gamma^{t}_{sk} + \Gamma^{s}_{ik}\Gamma^{t}_{st} \right)$$

das Variationsprinzip ohne jede Modifikation oder Nebenbedingung bildet, so erhält man als Gleichungssystem

$$\left.\begin{aligned} &R_{ik} = 0 \\ \text{und} \quad &\left(\mathfrak{V}^{ik}{}_{l} \equiv\right) \mathfrak{g}^{ik}{}_{,l} - \mathfrak{g}^{it}{}_{,t}\delta_l^k + \mathfrak{g}^{ik}\Gamma_l + \mathfrak{g}^{it}\Gamma_t\delta_l^k = 0 \end{aligned}\right\} (I)$$

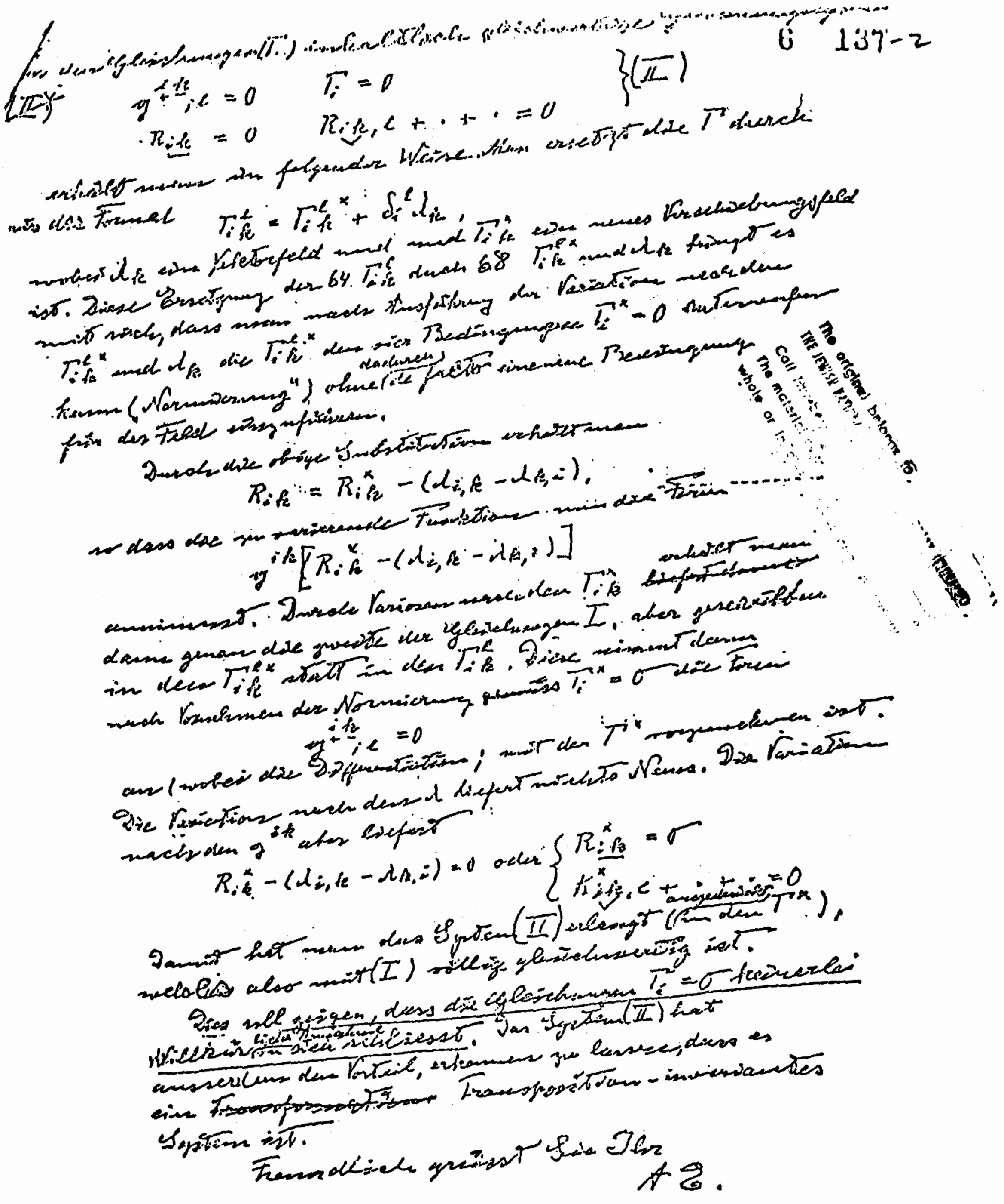
~~Auch~~ Hier kommt die Bedingung Γ_i nicht vor. Man kann die zweite Gleichung durch Eliminieren des zweiten Gliedes auch auf die Form bringen

$$-\mathfrak{g}^{\underline{ik}}{}_{,l} + \left(\mathfrak{g}^{ik}\Gamma_l - \frac{2}{3}\mathfrak{g}^{it}\Gamma_t\delta_l^k\right) = 0$$

Die Γ_l verschwinden hier also nicht.

Die Gleichungen in der von ~~mir~~ angegebenen Form sind lediglich eine Umformung von (I) ohne Aenderung des materiellen Gehaltes. Erst in diesen umgeformten Gleichungen tritt die Bedingung $\Gamma_i = 0$ und die Abschwächung der ersten Gleichung in die Form $R_{\underline{ik}} = 0$ $R_{\underset{\vee}{ik},l} + \cdot + \cdot = 0$ auf.

Dass die Gleichungen (I) noch einer Umformung bedürfen geht schon daraus hervor, dass ~~von~~ die zweite der Gleichungen (I) die Γ aus den $\mathfrak{g}$ nicht völlig bestimmt, indem diese Gleichungen also die Identität $\mathfrak{V}^{sk}{}_{,s,k} \equiv 0$ erfüllen, welche neben den vier Bianchi-Identitäten bestehen. Es bleibt daher von den Γ eine Funktion (von 4 Variablen) unbestimmt,



für die Gleichungen (I.) [illegible] gleichwertige [illegible]

$$\left.\begin{array}{ll} g^{\underset{+}{i}\underset{-}{k}}{}_{;l} = 0 & \Gamma_i = 0 \\ R_{\underline{ik}} = 0 & R_{\underset{\vee}{ik},l} + \cdot + \cdot = 0 \end{array}\right\}(II)$$

erhält man in folgender Weise. Man ersetzt die Γ durch mittels der Formel $\Gamma_{ik}^{l} = \Gamma_{ik}^{l*} + \delta_i^l \lambda_k$, wobei λ_k ein Vektorfeld und Γ_{ik}^{l*} ein neues Verschiebungsfeld ist. Diese Ersetzung der 64 Γ_{ik}^{l} durch 68 Γ_{ik}^{l*} und λ_k bringt es mit sich, dass man nach Ausführung der Variation nach den Γ_{ik}^{l*} und λ_k die Γ_{ik}^{l*} den vier Bedingungen $\Gamma_i^* = 0$ unterwerfen kann („Normierung") ohne dadurch ~~factor~~ eine neue Bestimmung für das Feld einzuführen.

Durch die obige Substitution erhält man

$$R_{ik} = R_{ik}^* - (\lambda_{i,k} - \lambda_{k,i}),$$

so dass die zu variierende Funktion ~~[illegible]~~ die Form

$$g^{ik}\left[R_{ik}^* - (\lambda_{i,k} - \lambda_{k,i})\right]$$

annimmt. Durch Variieren nach den Γ_{ik}^{*} erhält man ~~liefert dann~~ aber geschrieben dann genau die zweite der Gleichungen I, in den Γ_{ik}^{l*} statt in den Γ_{ik}^{l}. Diese nimmt dann nach Vornehmen der Normierung gemäss $\Gamma_i^* = 0$ die Form

$$g^{\underset{+}{i}\underset{-}{k}}{}_{;l} = 0$$

an (wobei die Differentiation; mit den Γ^* vorzunehmen ist. Die Variation Die Variation nach den λ liefert nichts Neues. Die Variation nach den g^{ik} aber liefert

$$R_{ik}^* - (\lambda_{i,k} - \lambda_{k,i}) = 0 \text{ oder } \left\{\begin{array}{l} R_{\underline{ik}}^* = 0 \\ R_{\underset{\vee}{ik},l}^* + \cdot \cdot = 0 \end{array}\right.$$

(angewendet auf den Γ^*),

Damit hat man das System (II) erlangt, welches also mit (I) völlig gleichwertig ist.

Dies soll zeigen, dass die Gleichungen $\Gamma_i = 0$ keinerlei Willkür[illegible] neue Annahme in sich schliesst. Das System (II) hat ausserdem den Vorteil, erkennen zu lassen, dass es ein ~~Transformations~~ Transpositions-invariantes System ist.

Freundlich grüsst Sie Ihr

A. E.

Einstein's letter to Bose in Paris from Princeton, dated 22 October 1953, commenting on Bose's paper 'A Unitary Field Theory . . .' for *Le Journal de Physique*; by courtesy of The Jewish National and University Library, Jerusalem.

The remaining three papers are all on the equation

$$g_{\mu\nu,\alpha} - g_{\rho\nu}\Gamma^{\rho}_{\mu\alpha} - g_{\mu\rho}\Gamma^{\rho}_{\alpha\nu} = 0.$$

In the third paper Bose considered the above equation as first order partial differential equations for $g_{\mu\nu}$. Then the condition of integrability is $g_{\mu\nu,\alpha\beta} = g_{\mu\nu,\beta\alpha}$. This condition reduced to equations involving Γ's and $g_{\mu\nu}$ but free of the derivatives of $g_{\mu\nu}$. They were explicitly worked out by Bose. This was a more or less routine investigation and did not lead to any very impressive result.

In the last two papers in the list Bose was interested in solving the above equations, regarding them as coupled algebraic equations for Γ's, taking the $g_{\mu\nu}$'s as known. The problem is to find the solution not for particular coordinates but in a form which will hold in all coordinate systems. This was then a challenging problem especially in view of a remark by Schrödinger (1947) :

> In the general case it is next to impossible to give a tensorial solution of these equations.

The adjective 'tensorial' in Schrödinger's remark is not quite correct for Γ's are themselves not tensors ; what he meant was that the solution must hold in all coordinate systems. However, Schrödinger was soon proved to be unduly pessimistic. Mme Tonnelat, Hlavaty in 1953, Bose, Einstein and Kaufman in 1954 and later on many workers including Mishra gave solutions. Although Bose's paper was published somewhat later than the first papers of Tonnelat and Hlavaty, Bose's first paper was communicated in September 1952, earlier than the publications of Tonnelat and Hlavaty. Indeed, all three were ignorant of the works of others and their methods were also different. Again, while Einstein had received a pre–publication manuscript from Bose and was aware of the work of Hlavaty, he and Kaufman proceeded without splitting $g_{\mu\nu}$ and Γ^{i}_{kl} into their symmetric and anti-symmetric parts, unlike the procedure adopted by Bose and others. Strangely, some later workers like Mavrides, Kichenssamy and even Mishra (1976) comment on the works of Tonnelat and Hlavaty, but they seem to ignore the works of Bose and also of Einstein and Kaufman.

IMPORTANCE OR MOTIVATION OF THESE WORKS

One may wonder what one can gain by these solutions. The first answer is that a pure mathematician's motivation arises simply from the mental desire to solve a problem just as the mountaineer climbs the Everest simply because 'it is there'.

Perhaps some of the eminent mathematicians had the idea that once the Γ's are found in terms of $g_{\mu\nu}$, these values of Γ would be plugged into the expression for $R_{\mu\nu}$ and one could attempt a general solution of the equation $R_{\mu\nu} = 0$. However, even in the much simpler case of general relativity, such a general solution of $R_{\mu\nu} = 0$ cannot be obtained and in reality has not even been attempted. What, therefore, one has to do is either to assume some symmetry and in that case the calculation of Γ's becomes not that difficult (and has been done by Bonnor, Papapetrou and others), or to deduce some general theorems. Thus to Einstein the crucial problem was: 'Do the singularity-free solutions of the equation system have physical meaning ? Are there at all singularity-free solutions which correspond to the atomistic character of matter and radiation ?' From this viewpoint the solution of those equations is not of great help. [Here those equations refer to the equations whose solution Bose presented in a letter to Einstein and we have just now considered.]

But, if the solution of the Γ's was not important from the point of view of the theory, one may ask: why did then Einstein and Kaufman give a solution ? The fact is, it came as a by-product of their investigation of the following problem. The universe we observe has three space-like and one time-like dimension. So, in order to be physically acceptable, the signature must be 2 or, in other words, the metric tensor determinant must be negative. Einstein and Kaufman considered provisionally $g_{\underline{\mu\nu}}$ to be the metric tensor and then investigated if this has the correct signature — do the field equations ensure that this signature will be maintained everywhere ? Their answer was in the affirmative, and in course of the proof they found the solution of the equations $g_{ik,l} = 0$.

As we come to the close of our discussion of Bose's works in unified field theories, some disturbing questions come to mind. Did Bose share Einstein's attitude towards quantum mechanics ? Did Bose believe that a viable unified field theory based on geometry can really come about while most physicists thought these attempts were futile ? Bose has left nothing in writing which can throw any light on these matters. In fact, his papers were purely mathematical exercises without any reference whatsoever to physics.

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10

Thermoluminescence

A Report on the Study of Thermoluminescence (1955)

Thermoluminescence means the phenomenon of emission of light (infrared or ultraviolet radiation included) other than pure thermal radiation, by a system under thermal stimulation. It is evidently related to energy states of the physical system, the relative transition probabilities and related electronic processes. When a system is excited by any method and a part of the excitation energy is stored in it, the system is termed thermoluminescent if it releases, on heating, a part or whole of the stored energy in the radiant form. The phenomenon of thermoluminescence exhibits a very large variety of behaviours, corresponding to different substances with variations in impurity contents or other imperfections. Thus, it is not surprising that thermoluminescence in all its details is a comparatively poorly understood phenomenon even in the case of simple materials.

Thermoluminescence appears to have been first observed by Boyle about three hundred years ago. Since the publication of Randall and Wilkins' work in 1945 there has been a steady accumulation of data in this field as well as attempts at theoretical interpretation of the experimental results in the light of the modern theory of solids, which already met with striking success in allied fields. The applicational possibilities of thermoluminescence in geological and archaeological work, dosimetry, or as a research tool in science and industry, have been already demonstrated by the tremendous growth in the number and variety of work in this field in the last fifty years.

Although extensive work has been carried out with thermoluminescence as a research tool, most of such work consists of the determination of trap depths and their changes under varying experimental conditions, but the results are far from satisfactory for a clear understanding of the electronic processes occurring inside the solid. The identification of the trapping and emitting centres, responsible for the different glow peaks, remains as yet a challenging problem in most of the phosphors. It was felt by Bose (and other pioneers in the field) that the physical interpretation of the results of investigations on thermoluminescence would be facilitated if the spectral compositions of the thermoluminescence and afterglow emission could be compared with the luminescence of the phosphor under study. The research workers in this field had been trying to find out the spectral compositions of thermoluminescence and afterglow emission with the help of filters; naturally, the results were indicative of the fact that the spectral composition of the emission undergoes changes during the process of thermoluminescence or afterglow decay in many cases but the data were not of convincing precision.

The duration of thermoluminescence is generally of the order of a minute and afterglow emission, comparatively poor in intensity, may also change in spectral composition as the intensity changes with time. The difficulties of interpretative work were thus essentially due to the lack of a suitable measuring device and experimental techniques. Bose saw the crux of the problem, and designed a rapid scanning spectrophotometer of comparatively high sensitivity to meet the requirements of the experimental workers in this field. The design and fabrication of the scanning spectrophotometer was reported by Bose at the International Conference on Crystallography held in Paris in 1954.

The spectrophotometer is capable of scanning the entire spectral range of visible and near-ultraviolet radiation in 0.9 seconds, and the spectral distribution of emission either in afterglow decay or thermoluminescence can be displayed on the screen of a cathoderay oscilloscope. It was thus possible to observe the changes in the spectral composition of emission during any time-varying process like thermoluminescence. The instrument performs the dual role of a spectrograph and a microphotometer, and can very conveniently be adapted to carry on studies of (i) fluorescence and phosphorescence, (ii) thermoluminescence spectra, (iii) the temperature dependence of luminescence (both fluorescence and phosphorescence), (iv) the correlation of colour centres and other known trapping centres with luminescence, (v) the development of emission centres in the phosphor during X-ray and cathoderay irradiation, and (iv) near-infrared absorption and emission spectra etc. The possibilities of the scanning spectrophotometer have however not been fully exploited till date.

The special features of the work initiated by Bose at the Khaira Laboratory in Physics in Calcutta were the luminescent and related properties of solids and frozen liquids under excitation by low energy electron beams as well as X-rays. The excitation is limited to a thin surface layer in the case of irradiation by low energy electrons, while for X-rays or other high energy excitations the entire volume of the crystals is affected. The two methods of excitation should thus bring out the distinctive differences in the effects of impurities , self-absorption and energy transfer processes, on the luminescent and allied properties of solids. Some work was also attempted on the luminescence of solids under soft X-ray excitation; and attempts were made to correlate the data on valence band spectra with those on colour centres and luminescence of simple solids.

Adapted from H N Bose, *On the Pioneering Works of Professor S N Bose in the Field of Thermoluminescence,* in the Proceedings of the Seminar on The Scientific Contributions of Professor S N Bose, Cal Math Soc 1943, pp. 111-121.

11

Bose Statistics : a historical perspective

a. Planck's derivation of his law of black-body radiation

In order to appreciate the novelty and importance of Bose's work in its historical perspective, it is necessary first to give a brief account of Planck's original derivation [Planck, 14 December 1900] of the law of black-body radiation,

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \tag{1}$$

which he had discovered empirically a few weeks earlier (7 October 1900). It consisted of *three* steps.

First, he established the relation

$$\rho_\nu = \frac{8\pi\nu^2}{c^3} U_\nu \tag{2}$$

between the energy density ρ_ν of incident radiation at temperature T, whose frequency lies between ν and $\nu + d\nu$, and the average energy U_ν of a resonator of the same frequency ν at temperature T, on the basis of *classical* electromagnetic theory. Comparing (1) and (2), he found U_ν :

$$U_\nu = \frac{h\nu}{e^{h\nu/kT} - 1} . \tag{3}$$

In the second step he determined the entropy of these oscillators by integrating $TdS=dU$ where T is taken from (3) as a function of U (for fixed ν). He obtained

$$S = k\left[\left(1+\frac{U_\nu}{h\nu}\right)\ln\left(1+\frac{U_\nu}{h\nu}\right) - \frac{U_\nu}{h\nu}\ln\frac{U_\nu}{h\nu}\right]. \tag{4}$$

The third step was the revolutionary one. He introduced two ideas at this stage that he himself considered as 'acts of desperation'. He assumed that the total energy $U_N = N\,U_\nu$ of N oscillators (resonators) was made up of finite energy elements ϵ such that $U_N = P\epsilon$ with P a large number. Then he searched in Boltzmann's work for a permutation measure W_N (the total number of complexions or distributions) of P discrete equal energy values ϵ among N oscillators that would correspond to the right hand side of (4) when used in $S_N = k\ ln\ W_N$. He found

$$W_N = \frac{(P+N-1)!}{P!\,(N-1)!}\,.. \tag{5}$$

Using $P/N = U_\nu/\epsilon$, $S_N = N\,S$ and applying Stirling's formula, he obtained

$$S = k\left[\left(1+\frac{U_\nu}{\epsilon}\right) ln \left(1+\frac{U_\nu}{\epsilon}\right) - \frac{U_\nu}{\epsilon}\, ln\, \frac{U_\nu}{\epsilon}\right]. \tag{6}$$

Since S is a function of (U_ν/ν) only, it follows from (4) and (6) that

$$\epsilon = h\,\nu\,.$$

This is how quantum theory was born. Planck had no more justification for using the formula (5) than that it gave him the result that he was after : '. . . a theoretical interpretation had to be found at any price, however high it might be,' he wrote to Robert William Wood in October 1931. [Mehra and Rechenberg, 1982, **1**, *1*, 50]. Actually, Planck's combinatorial approach differed from Boltzmann's probabilistic method in that Planck associated W_N with S_N, the equilibrium entropy, without maximizing it. For Boltzmann W_N was the number of possible complexions corresponding to the macrostate which can be realized by the largest number of complexions. As pointed out by Ehrenfest, a strict adherence to the accepted principles of statistical mechanics would have led Planck to the classical law of Rayleigh and Jeans! [Ehrenfest, 1905]

b. Einstein's light-quantum hypothesis

In 1905 Einstein argued on the basis of his analysis of energy fluctuations of radiation obeying Wien's law that such radiation had statistical properties similar to material particles and so must consist of discrete light-quanta of magnitude $(R/N_o)\,\beta\nu = h\nu$. [Einstein, 1905] This light-quantum hypothesis was immediately applied by him to give a reasonable explanation of several radiation phenomena, specially the photo-electric effect. Since these conclusions contradicted the classical electromagnetic theory on which Planck attempted to base his theory of heat radiation, Einstein considered

Planck's theory 'in some way a counterpart' to his own theory. He subjected Planck's derivation to a critical analysis and came to the following conclusion :

> We must therefore regard the following law as the basis of Planck's quantum theory of radiation : the energy of an elementary resonator can only assume values which are integral multiples of $(R/N_o)\,\beta\nu$; the energy of a resonator changes in jumps by absorption or emission in integral multiples of $(R/N_o)\,\beta\nu$. . . . If the energy of a resonator can alter only in jumps, then for the evaluation of the average energy of a resonator in a radiation cavity, the usual [electromagnetic] theory cannot be used, for the latter does not admit any distinctive energy values for a resonator. [Einstein, 1906]

Two years later Lorentz also came to the same conclusion [Lorentz to Wien, 6 June 1908; Mehra and Rechenberg, 1982, **1**, *1*, 98]. Planck attempted to modify his theory to take into account the quantum of action *without violating any aspect of classical electrodynamics*, but it gradually became clear from his studies and those of others that it was impossible to establish the theory of black-body radiation entirely on the foundation of Maxwell's classical electrodynamics and the statistical mechanics of Maxwell and Boltzmann.

c. Debye's derivation of Planck's law

In 1910 Debye gave a new derivation of Planck's radiation law in which he sought to avoid the inconsistencies (as viewed from Maxwell's theory) of Planck's earlier derivations as pointed out by Einstein [Debye, 1910]. Instead of using the relation (2) between the radiation density ρ_ν and the average energy of the oscillator U_ν, Debye calculated 'the probability for a given state of radiation and therefore as is well known, the entropy, using the properties of the state alone without employing resonators.' In agreement with Rayleigh and Jeans, he calculated the number $N_\nu\, d\nu$ of elementary states or vibrational modes (*Hohlraum oscillators*) contained in a volume V and obtained

$$N_\nu\, d\nu = \frac{8\pi\, \nu^2\, V\, d\nu}{c^3} . \tag{8}$$

Assuming that an amount of energy $h\nu$ gets distributed over each of these vibrations according to an arbitrary distribution function f_ν, he obtained

$$\rho_\nu\, d\nu = \frac{8\pi h \nu^3}{c^3}\, f_\nu\, d\nu . \tag{9}$$

Now, 'analogously to Planck', Debye *assumed* that the probability of distributing $N_\nu f_\nu\, d\nu$ quanta of energy $h\nu$ among $N_\nu\, d\nu$ vibrations was given by

$$\omega_\nu = \frac{(N_\nu\, d\nu + N_\nu f_\nu\, d\nu)\,!}{(N_\nu\, d\nu)\,!\,\,(N_\nu f_\nu\, d\nu)\,!} . \tag{10}$$

He then calculated the distribution function f_ν in the following way. He calculated the maximum of ω_ν with the constraint that the energy remained constant. From this he calculated the equilibrium entropy S_ν to be $k\ ln\ (\omega_\nu^{max})$. Finally, using the definition of entropy to be $(dS_\nu/d\rho_\nu) = T^{-1}$, he obtained

$$f_\nu = \frac{1}{e^{h\nu/kT} - 1}\,. \tag{11}$$

Together with (9), this yielded Planck's radiation formula (1).

Two important features of Debye's derivation need to be emphasized as background to Bose's derivation that came fourteen years later. First, it became clear from his derivation that Planck's law follows simply from the assumption that the energy transfer from matter to the heat radiation and vice versa is quantized in units of $h\nu$ and that *no knowledge of the properties of material resonators is needed.* In other words, the quantum hypothesis only determined the transfer of energy from one mode of the radiation to another. This was in keeping with Planck's ideas but not Einstein's. Second, he used Planck's definition of the probabilities ω_ν, without analyzing its significance.

d. Indistinguishability of Planck's quanta

In 1911 Ladislas Natanson [1911] subjected the Planck-Debye combinatorial procedure to a critical analysis and showed that it was equivalent to distributing *P indistinguishable* energy elements ϵ among *N* 'receptacles of energy' distinguished only by the numbers *j* of quanta in them so that N_j receptacles contain *j* quanta each, subject to the constraints $\sum_j N_j = N$ and $\sum_j j N_j = P$. He found

$$\omega_\nu = \frac{(N_\nu\, d\nu)\,!}{\prod_{j=0}^{p} N_j\,!}\,. \tag{12}$$

Contrary to popular belief, therefore, the *indistinguishability of the quanta had already been tacitly assumed by Planck and Debye and this was first noticed by Natanson already in 1911.*

Ehrenfest and Kamerlingh Onnes [1914] also came to a similar conclusion in 1914. They however claimed that Planck's procedure was equivalent to distributing *P indistinguishable* energy elements ϵ among *N distinguishable* resonators and arrived at the expression

$$\omega_\nu = \frac{(N-1+P)\,!}{P\,!\,(N-1)\,!} \tag{13}$$

which is equivalent to the expressions (10) and (12) when N>>1. They then proceeded to clarify the distinction between Planck's energy quanta *which were statistically not independent* (they called them 'non-disjointed quanta') and Einstein's hypothetical light-quanta which were believed to be *statistically independent of one another* ('disjointed quanta'). The concluded : *'Planck's formal device* (distribution of P energy-elements $\in$ among N resonators) *cannot be interpreted in the sense of Einstein's light-quanta.'*

e. The introduction of phase space cells by Planck

Planck himself had come to the view on his own that classical statistical mechanics had to be modified in order to yield his radiation formula (1) rather than that of Rayleigh and Jeans. He reported his important conclusion at the Solvay Congress held in Brussels in 1911. According to Gibbs, the probability of finding a single particle in the element $d^3p\, d^3q$ of the six-dimensional phase space is given by

$$\frac{e^{-E/kT}\; d^3p\, d^3q}{\int e^{-E/kT}\; d^3p\, d^3q} \tag{14}$$

where $E = E(p,q)$ is the energy of the system. For a classical one-dimensional oscillator, $E = p^2/2m + \beta q^2/2$ and so the average energy of such an oscillator is kT. However, if E can take only the discrete values $E_n = n \in = n h \nu$ with n=0,1,2 . . ., then

$$U = \frac{\sum\limits_n E_n\, e^{-E_n/kT}}{\sum\limits_n e^{-E_n/kT}} = \frac{\in}{e^{\in/kT} - 1}, \tag{15}$$

in agreement with (3). It was this straightforward derivation that led Planck for the first time to make the explicit statement that the energy of an oscillator (and not only its average energy) was an integral multiple of $h\nu$. It also led him to interpret the constant *h as a finite extension of the elementary area in phase space.* The motions of one-dimensional classical oscillators can be described by a family of concentric ellipses of semi-axes $(2E/\beta)^{1/2}$ and $(2\,m\,E)^{1/2}$. Since these oscillators can have any energy E, the ellipses form a continuum. However, if it is assumed that these ellipses are separated from each other so that the nth ellipse encloses an area

$$\int\int dp\,dq = n\,h\,, \tag{16}$$

then the energies of the oscillators must be integral multiples of $h\nu$. Planck therefore drew the revolutionary conclusion that energy quanta were a consequence of the fundamental condition

$$\int_E^{E+\in}\int dp\,dq = h\,. \tag{17}$$

This implied a revision of the basic tenets of classical mechanics. He declared : 'The framework of classical mechanics, even if combined with the Lorentz-Einstein principle of relativity, is obviously too narrow to account for all those physical phenomena which are not directly accessible to our coarse senses . . . One should therefore confine oneself to the principle that the elementary region of probability h has an ascertainable finite value and avoid all further speculation about the physical significance of this remarkable constant.' In other words, all attempts at finding a classical mechanical explanation of h must be abandoned. As we shall see later, to Bose these considerations (with which he was familiar) brought home extremely important lessons that would influence his own seminal contribution to the subject. First, he took serious note of the conclusion that classical electrodynamics was essentially incompatible with Planck's radiation formula. Second, since the quantum states of a particle were not continuously distributed throughout phase space (as in classical dynamics), he concluded that their number could be counted by simply dividing the total volume of phase space by h^3, the volume of an elementary cell. Finally, he took seriously Planck's claim that classical statistical mechanics had to be modified in a special way in order to conform to the requirements of quantum theory.

f. Spontaneous and induced transitions

The next important step was taken by Einstein in 1916. In 1913 Niels Bohr had proposed his model of the atom with stationary states with discrete energy values [Bohr,1913, b,c]. The transitions between any pair of these states were assumed to be caused by the exchange of energy quanta $h\nu$ between the atom and the radiation field described by *classical electrodynamics*. Einstein gave an elegant derivation of Planck's radiation formula by considering Bohr atoms in thermal equilibrium with radiation in a cavity but *without making any use of classical electrodynamics* [Einstein, 1916a]. He used Boltzmann's principle to write the probability W_n for an atom to be in a stationary state with quantum-number n in the form

$$W_n = g_n \, exp\left(-\frac{\epsilon_n}{k\,T}\right), \tag{18}$$

g_n being the statistical weight of the state. He then assumed that a stationary state n may pass to a stationary state m of higher energy ($\epsilon_m > \epsilon_n$) by absorbing a light-quantum of frequency $\nu_{n\,m}$, the rate of transition being $B_n^m\, N_n\, \rho_\nu$ (N_n being the number of atoms in the stationary state n and B_n^m a proportionality constant characterizing the absorption of radiation of frequency $\nu_{n\,m}$ in the presence of external radiation of density ρ_ν). Transitions from a stationary state m of higher energy to a stationary state n of lower energy by the emission of a light-quantum of frequency $\nu_{m\,n} = \nu_{n\,m}$ could, however, take place in *two* ways. Atoms can emit this radiation *independent* of the external field, their number being $A_m^n\; N_m$ (A_m^n being the constant of proportionality). About this Einstein said : 'One can hardly think of it in any other way except as a radioactive reaction.' It came to be known later as 'spontaneous emission.' Atoms can also be *induced* to emit radiation by the external field, the rate being $B_m^n\; N_m\; \rho_\nu$. Einstein called this 'negative radiation'. In thermal equilibrium one must have $(A^n_m N_m + B^n_m N_m \rho_\nu) = B^m_n N_n \rho_\nu$. Using (18) to calculate N_n/N_m, one gets

$$A^n_m \, g_m + \rho_\nu \left[B^m_n \, g_n \exp\left(\frac{\epsilon_m - \epsilon_n}{kT}\right) - B^n_m \, g_m \right]. \tag{19}$$

At this stage Einstein had to take his guidance from classical theory by essentially applying the 'correspondence principle' according to which classical theory should be a limiting case of quantum theory. In the classical limit ρ_ν goes to infinity as the temperature T goes to infinity for fixed ν (Rayleigh-Jeans law, $\rho_\nu \alpha T$). This gave $B^n_m g_m = B^m_n g_n$. With this condition (19) could be written as

$$\rho_\nu = \frac{A^n_m / B^n_m}{\exp\left[(\epsilon_m - \epsilon_n) / kT\right] - 1}. \tag{20}$$

Then Einstein used Wien's displacement law to infer that A^n_m/B^n_m must be proportional to ν^3, and the Rayleigh-Jeans law in the limit of low frequencies to determine the constant of proportionality to be $8\pi h/c^3$. Thus he obtained the relation

$$A^n_m = \frac{8 \pi h \nu^3}{c^3} B^n_m \tag{21}$$

and

$$\epsilon_m - \epsilon_n = h\nu \tag{22}$$

which is Bohr's frequency condition. These two relations when substituted into (20) gave Planck's law. This is a remarkable derivation that uses an admixture of the light-quantum hypothesis and Bohr's atomic theory which Bohr regarded as being imcompatible, Wien's radiation law and therefore implicitly the Maxwell-Boltzman distribution for the light-quanta, and the correspondence principle which was a heuristic principle without any fundamental basis. Einstein concluded by saying : 'I admit freely, of course, that the three hypotheses concerning outgoing and incoming radiation, do not at all become substantiated results by the mere fact that they lead to Planck's radiation formula. But the simplicitly of the hypotheses, the generality with which the consideration can be carried through easily, as well as the natural connection of the consideration employed with the limiting case of Planck's linear oscillator (in the sense of classical electrodynamics and mechanics), persuaded me to regard it as very probable that all this constitutes the fundamental outline of the future theoretical derivation.' Although the hypotheses concerning the outgoing and incoming radiation turned out to be correct to a large extent (though not entirely so, as we shall see later), the future derivation (given by Bose) did not make use of them at all. In fact, they became the major bone of contention between him and Bose.

By 1916 the light-quantum hypothesis had received a measure of empirical support from Millikan's verification of Einstein's photo-electric equation. Millikan's paper [1916b] appeared in the March 1916 issue of the *Physical Review*. Einstein quickly realized from his new derivation of Planck's law that atoms seemed to interact with radiation as if single atoms collided with light-quanta of energy $h\nu$ and directed momentum $h\nu/c$. He wrote to his friend, Michael Besso :

> The fundamental thing is that the *statistical* consideration, which leads to Planck's formula, has become *unified* and thereby as general as one can imagine, since one has not assumed anything about the nature of the mediating molecules other than the most general quantum idea. From it follows the result (which was not contained in the paper I sent you) that in each energy transfer from radiation to matter the momentum $h\nu/c$ is also transferred to the molecule. Hence we conclude that every such elementary process is a *completely directed* event. With that the light-quanta must be considered as being substantiated. [Einstein to Besso, 6 September 1916; Mehra and Rechenberg, 1982, **1**, *2*, 515]

However, Einstein was not really satisfied. In a subsequent paper he remarked :

> The weakness of the theory lies, on one hand, in the fact that it does not bring us closer to a connection with the undulatory theory, and, on the other hand, in the fact that it leaves the instant and direction of the elementary processes to chance. [Einstein, 1916b, 1917]

g. Reactions to the light-quantum hypothesis: discovery of the Compton effect

The majority of physicists, in fact, did not take the light-quantum hypothesis seriously, even after Millikan's work. Millikan himself remarked in his 1916 paper on the photo-electric effect : 'Yet the semi-corpuscular theory by which Einstein arrived at this equation seems at present to be wholly untenable.' The main difficulty lay, as Einstein himself realized, in accounting reasonably for the interference and diffraction phenomena observed with all kinds of radiation in terms of light-quanta. Evidence in favour of the wave theory was overwhelming and mounting. In 1912 Max von Laue, Walter Friedrich and Paul Knipping had settled the issue in favour of X-rays being electromagnetic waves of extremely short wavelengths through the phenomenon of interference of X-rays in crystals [von Laue, Friedrich and Knipping, 1912]. In 1913 Planck, Nernst, Rubens and Warburg who wished to bring Einstein to Berlin, wrote to the Prussian Ministry of Education:

> That he may sometimes have missed the target in his speculations, as, for example, in his theory of light-quanta, cannot really be held against him. For in the most exact of natural sciences every innovation entails risk. [Clark, 1972, 215]

Bohr had his own reasons for not favouring the light-quantum hypothesis though he realized from the beginning that the laws of electrodynamics had to be abandoned for a proper description of quantum phenomena. The hypothesis ran counter to his plans to build atomic theory on the basis of the correspondence principle.

The general acceptance of the light-quantum hypothesis eventually came in 1923 through the work of Debye [1923] and Compton [1923]. Compton realized that the shift in the wavelength of X-rays scattered by atoms that he observed could not be understood in terms of the classical wave theory of scattering but only in terms of elementary processes of scattering of light-quanta by the electrons in atoms, in which both energy and momentum were conserved. Debye also arrived at the same conclusion independently. Arnold Sommerfeld who was then visiting the University of Wisconsin as Carl Schurz Professor of Physics came to know about Compton's results. On 21 January 1923 he wrote to Bohr : 'The most interesting scientific news I have learned in America is a work of Arthur Compton in St. Louis. According to it the wave theory of X-rays would finally have to be given up.' [Mehra and Rechenberg, **1**, *2*, 529] Wherever he went, Sommerfeld referred to Compton's work and its importance for the quantum theory. On his return to Germany he continued to advocate Compton's work. Several months later he wrote to Compton : 'Your discovery of the change in wavelength of X-rays also occupies the scientific community in Germany most deeply. I met Einstein and Kossel in August, and we mainly discussed your effect.' [Sommerfeld to Compton, October 1923; Mehra and Rechenberg, 1982, **1**, *2*, 530]

h. Pauli's derivation of Planck's law

Wolfgang Pauli came to know of Compton's discovery in the summer of 1923 when he came to Hamburg on several weeks' leave of absence from Copenhagen. He immediately got interested in the problem of thermal equilibrium between radiation and free electrons. Lorentz had reported in the first Solvay Conference that he could not establish the condition of thermal equilibrium for electrons having a Maxwellian velocity distribution [Lorentz's report in Langevin and de Broglie, 1912]. Pauli wished to re-examine the problem by considering reversible collisions between electrons and directed light-quanta, obeying the laws of conservation of energy and momentum. Pauli [1923] was able to show that thermal equilibrium characterized by Planck's law was possible for electrons having a Maxwellian distribution provided the probability of collisions per unit time was given by

$$A\,\rho_\nu + B\,\rho_\nu\,\rho_{\nu'}\,, \tag{23}$$

where A and B are the coefficients of Einstein's radiation theory [Einstein, 1917] and ρ_ν and $\rho_{\nu'}$ the densities of radiation before and after the collision respectively. Pauli showed that the first term was dominant for radiation satisfying Wien's law while the second term was more important for radiation of longer wavelengths satisfying the Rayleigh-Jeans law. Pauli interpreted this as the effect of interference fluctuations of classical waves. Einstein and Ehrenfest welcomed Pauli's work and clarified and generalized it to cases in which the elementary processes involved more than two light-quanta [Einstein and Ehrenfest, 1923]. In particular, they discussed the 'paradoxical' second term in (23) 'which implies that the number of the elementary processes of scattering occurring per unit time at an electron (which is at quasi rest) increases faster than being proportional to the radiation density ρ' of frequencies ν', which the radiation quantum possesses after being modified by the elementary process.' They concluded that this term followed from a detailed considertion of all interaction processes between radiation and electrons, including the induced emission of light (Einstein's 'negative radiation').

i. Bose's derivation

While the leading European and American physicists were skeptical, even hostile to Einstein's light-quantum hypothesis, two unknown young physicists in India, M N Saha and S N Bose grasped its importance and started using it with great success. Within two years of Einstein's 1917 paper from which he drew the revolutionary conclusion that a light-quantum carried directed momentum $h\nu/c$, M N Saha, a close friend of Bose, applied it to explain the existence of radiation pressure on objects of molecular size [Saha, 1919]. On the basis of the classical theory Nicholson [1914] and Klotz [1918] had shown that radiation pressure on particles of molecular size should become totally 'evanescent'. Saha applied 'quantum theory in the place of the old continuous theory of light. Instead of assuming that "light" is spread continuously over all points of space,' he supposed that 'they are localized in pulses of energy $h\nu$ . . . Let this pulse encounter a molecule m and be absorbed by it. Then in doing so the molecule will be thrust forward with an impulsive momentum of $h\nu/c$, (c=velocity of light); for we may suppose the pulse to have the mass $h\nu/c^2$ and the momentum $h\nu/c$; the absorption of the pulse by the molecule may be taken as a case of inelastic impact, the whole momentum being communicated to the molecule. The velocity with which the molecule will move forward = $h\nu/mc$.' Saha concluded that *'radiation-pressure may exert an effect on the atoms and molecules which are out of all proportion to their actual sizes.'* This must have been the first application of Einstein's hypothesis that light-quanta carried a directed momentum $h\nu/c$.

In March 1924 Saha visited Dhaka and stayed with Bose. While teaching postgraduate students Bose had keenly felt the need for a derivation of Planck's law free of logical difficulties. Saha drew Bose's attention to the papers of Pauli [1923] and their connection to Einstein and Ehrenfest [1923] and their connection to Einstein's paper of 1917. In an interview with J Mehra [Mehra and Rechenberg, **1**, *2*, 565] Bose recalled : 'What seemed to be happening in Pauli's work was that in order to apply the quantum conditions you had to know exactly what was going to happen afterwards.' Stimulated by his discussions with Saha, Bose began to study carefully the works of Planck [1900,1910], Peter Debye [1910,1923], Einstein [1917], Arthur Compton [1923], Pauli [1923] and Einstein and Ehrenfest [1923]. This resulted in two papers that he wrote in June 1924 [Bose, 1924a, 1924b] and sent to Einstein for his views and for publication in *Zeitschrift für Physik.* Einstein translated both of them into German and had them published in *Zeitschrift für Physik* in 1924. To both of them he added his own comments.

It is clear from these two papers of Bose which he always regarded as integral parts of each other that he clearly recognized the following points :

(1) The fundamental assumptions of the quantum theory are incompatible with the laws of classical electrodynamics [Bose, 1924a].

(2) All previous derivations had made use of relation (2) 'between the density of radiation and the average energy of an oscillator, and they make assumptions

about the number of degrees of freedom of ether, which enters the above equation (the first factor on the right hand side). This factor could however be derived only from the classical theory. This is an unsatisfactory feature in all derivations and it is not surprising that efforts are made again and again to give a derivation free from this logical flaw.' [Bose, 1924a]

(3) Even Einstein's 1916/1917 derivation, although 'remarkably elegant', makes use of 'Wien's displacement law and Bohr's correspondence principle. Wien's law is based on classical theory and the correspondence principle assumes that the quantum theory agrees with the classical theory in certain limiting cases.' [Bose, 1924a].

(4) 'In all cases it appears to me,' wrote Bose [1924a] 'that the derivations are not sufficiently justified from a logical point of view. On the other hand, the light-quantum hypothesis combined with statistical mechanics *(as adapted by Planck to conform to the requirements of quantum theory)* appears sufficient for the deduction of the law independent of classical theory' (our italics). In other words, the incompatibility between Einstein's light-quanta ('disjointed quanta') and Planck's law (implying 'undisjointed quanta') can be removed by adapting the statistical mechanics of light-quanta to conform to the requirements of quantum theory, as proposed by Planck.

(5) 'The problem of thermodynamic equilibrium of radiation in the presence of material particles can however be studied using the methods of statistical mechanics, *independently of any special assumption about the mechanism of the elementary processes on which the energy exchange depends.'* [Bose, 1924b]

Bose therefore proceeded in four steps.

First step: He took Einstein's light-quantum hypothesis seriously and treated black-body radiation as a collection of light-quanta enclosed in a volume V. If there are N_s light-quanta of energy $h\nu_s$ ($s = 0$ $s = \infty$), the total energy E is

$$E = \sum_s N_s h \nu_s = V \int \rho_\nu d\nu \,. \tag{24}$$

ρ_ν can be determined from this simple relation if N_s is known. So, the problem is to determine N_s. 'If we can give the probability for each distribution characterized by arbitrary values of N_s, then the solution is determined by the condition that this probability is a maximum subject to the subsidiary condition' (24) [Bose,1924a]. Bose then proceeded to find this probability.

Second step: Before he could do that, he needed to compute the number of *states* of a light-quantum whose frequency lies between ν_s and $\nu_s + d\nu_s$. In order to be able to do this, he *extended* Planck's idea of dividing up the phase space of a material oscillator into elementary cells of volume h^3 *to the phase space of light-quanta.* In other words, *he extended Planck's 'first quantization' of material oscillators to the radiation field itself.* This was the most novel and crucial input. Emboldened by Compton's discovery, he assumed the momentum of a light-quantum of frequency ν to be $h\nu/c$ in the direction of its motion. He must have also been aware of Saha's success in explaining radiation pressure on molecular-size objects with the help of this hypothesis [Saha,1919]. The instantaneous *state* of such a light-quantum, he stated, is characterized by a point in six-dimensional phase space which is constrained to lie on the surface of a cylinder defined by

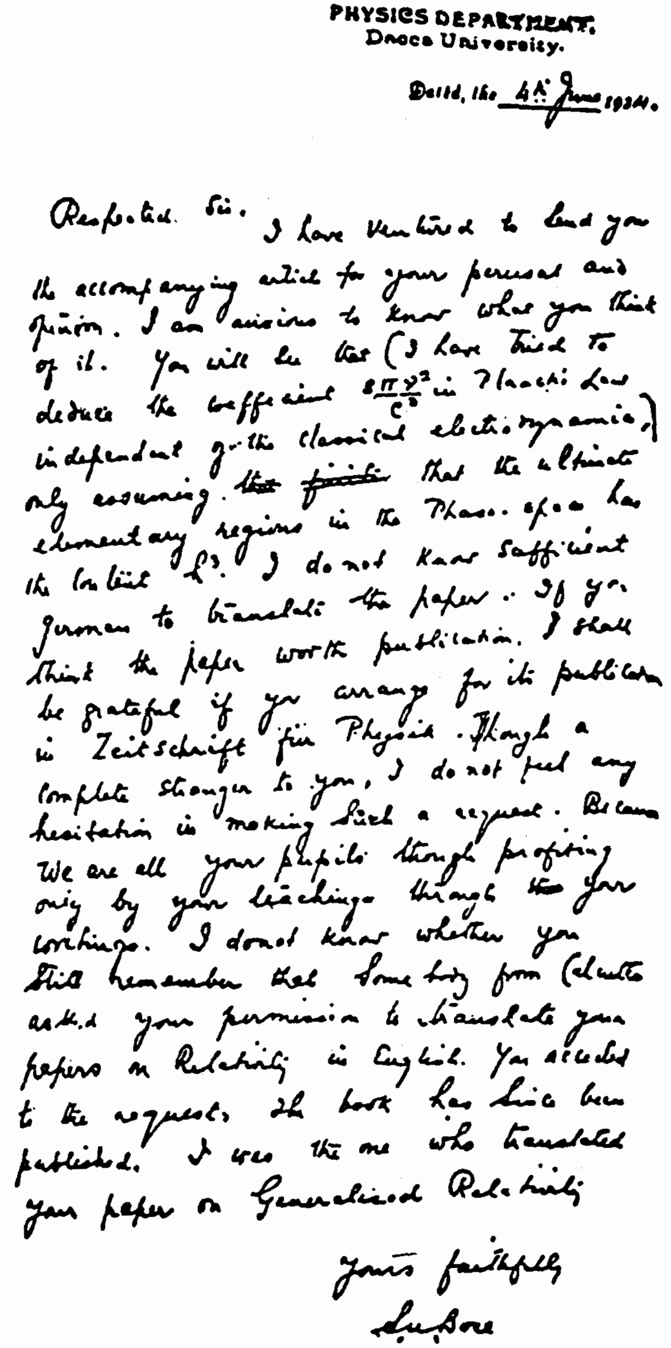
$$p_x^2 + p_y^2 + p_z^2 = \frac{h^2 \nu_s^2}{c^2} . \tag{25}$$

Then the phase space belonging to the frequency interval $d\nu_s$ is

$$\int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = 4\pi V \left(\frac{h\nu_s}{c}\right)^2 \frac{h d\nu_s}{c} = 4\pi \frac{h^3 \nu_s^2}{c^3} V d\nu_s . \tag{26}$$

Dividing this by h^3, the 'ultimate elementary region' [Bose to Einstein, 4 June 1924], Bose obtained $4\pi V\nu_s^2\, d\nu_s/c^3$. Then the following statement occurs : 'In order to take into account the *fact* of polarization, it seems however appropriate to multiply this number once again by 2' (our italics) to obtain $8\pi V\nu_s^2\, d\nu_s/c^3$ as the number of cells belonging to $d\nu_s$. No explanation is offered as to how this 'fact of polarization', an essentially *classical* concept, can be understood in terms of the light-quantum hypothesis, although Bose claimed to deduce this factor *independent of the classical electrodynamics* [Bose to Einstein, 4 June 1924]. Bose had always maintained privately that he did offer a quantum theoretic explanation but Einstein removed it from his translation and substituted it by the statement about the polarization factor. In his letter to Bose, Einstein simply wrote : 'You are the first person to derive the factor quantum theoretically, even though because of the polarization factor 2, not wholly rigorously' [Einstein to Bose, 2 July 1924]. What was the explanation that Bose claimed to have offered? It was that *light-quanta carried an intrinsic spin that could take only the values* $\pm h/2\pi$. There is only one recorded evidence of this because Bose's original manuscript in English is missing from the Einstein archives. It appears in a paper by C V Raman and S Bhagavantam [1931]. The paper is entitled 'Experimental Proof of the Photon Spin'. They write :

> In his well-known derivation of the Planck radiation formula from quantum statistics, Prof. S.N. Bose obtained an expression for the number of cells in phase space occupied by the radiation, and found himself obliged to multiply it by a numerical factor 2 in order to derive from it the correct number of possible arrangements of the quantum in unit volume. The paper as published did not contain a detailed discussion of the necessity for the introduction of this factor, but we understand from a personal communication by Prof. Bose that he envisaged the possibility of the quantum possessing besides energy $h\nu$ and momentum $h\nu/c$ also an intrinsic spin angular momentum $\pm h/2\pi$ round an axis parallel to the direction of its motion. The weight factor 2 thus arises from the possibility of the spin of the quantum



PHYSICS DEPARTMENT,
Dacca University.

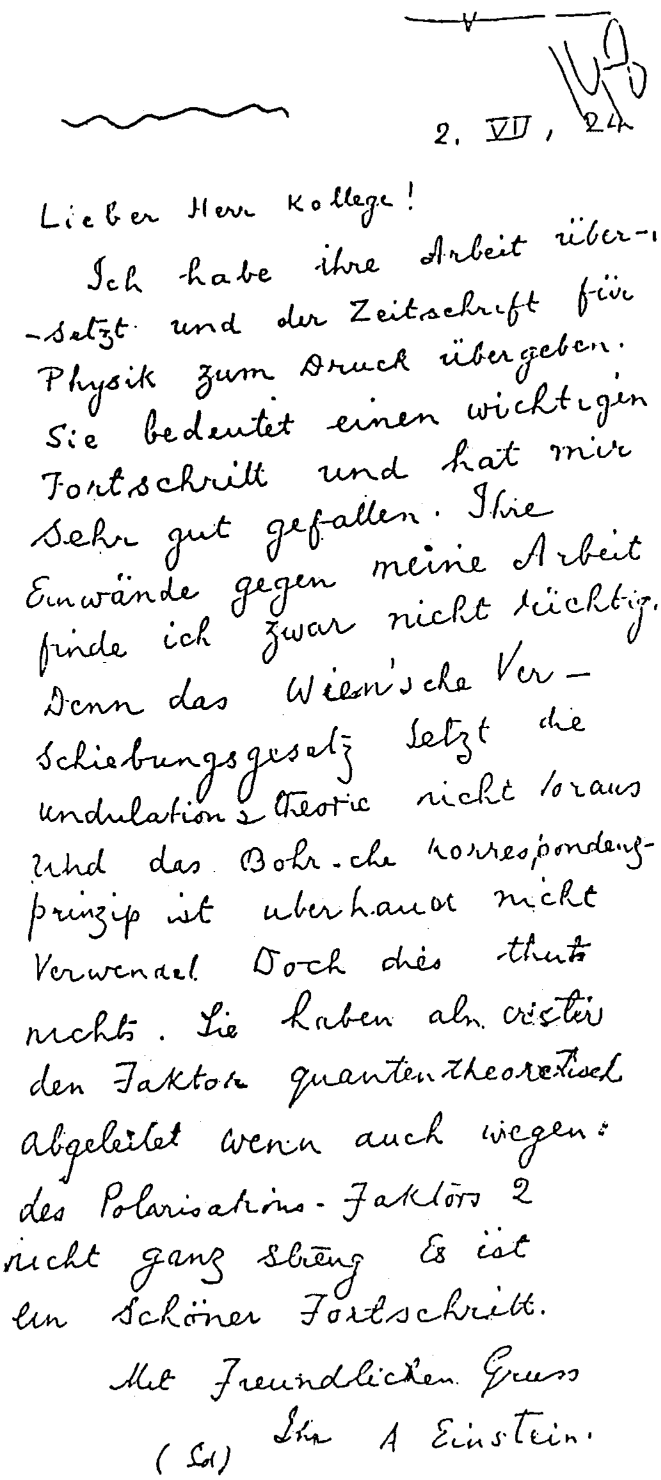
Dated, the 4th June 1924.

Respected Sir,

I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think of it. You will see that (I have tried to deduce the coefficient $\frac{8\pi\nu^2}{c^3}$ in Planck's Law independent of the classical electrodynamics,) only assuming ~~the finite~~ that the ultimate elementary regions in the Phase-space has the content h^3. I do not know sufficient German to translate the paper. If you think the paper worth publication, I shall be grateful if you arrange for its publication in Zeitschrift für Physik. Though a complete stranger to you, I do not feel any hesitation in making such a request. Because we are all your pupils though profiting only by your teachings through ~~the~~ your writings. I do not know whether you still remember that somebody from Calcutta asked your permission to translate your papers on Relativity in English. You acceded to the request. The book has since been published. I was the one who translated your paper on Generalised Relativity.

Yours faithfully
S. N. Bose

Bose's letter of 4 June 1924 to Einstein



2. VII, 24

Lieber Herr Kollege!

Ich habe ihre Arbeit über-
-setzt und der Zeitschrift für
Physik zum Druck übergeben.
Sie bedeutet einen wichtigen
Fortschritt und hat mir
sehr gut gefallen. Ihre
Einwände gegen meine Arbeit
finde ich zwar nicht richtig.
Denn das Wien'sche Ver-
schiebungsgesetz setzt die
undulations theorie nicht voraus
und das Bohr-che korrespondenz-
prinzip ist uberhauot nicht
verwendet. Doch dies thuts
nichts. Sie haben als erster
den Faktor quantentheoretisch
abgeleitet wenn auch wegen
des Polarisations-Faktors 2
nicht ganz streng. Es ist
ein schöner Fortschritt.

Mit freundlichen Gruss

Ihr A Einstein.
(Sd)

Einstein's postcard of 2 July 1924 to Bose

> being right-handed or left-handed, corresponding to the two alternative signs of the angular momentum. There is a fundamental difference between this idea, and the well-known result of classical electrodynamics to which attention was drawn by Poynting and more fully developed by Abraham that a beam of light may in certain circumstances possess angular momentum . . . Thus, according to the classical field theory, the angular momentum associated with a quantum of energy is not uniquely defined, while according to the view we are concerned with in the present paper, the photon has always an angular momentum having a definite numerical value of a Bohr unit with one or other of the two possible alternative signs.

Why did Einstein remove Bose's explanation? Probably he felt it was too radical and unnecessary at that stage. However, instead of removing it altogether, he could have added a suitable comment to the one he did make at the end of the paper. In fact, he did not hesitate to add a dissenting note to Bose's second paper [1924b] as we shall see later.

Be that as it may, let us proceed with Bose's derivation. Having divided the phase space volume belonging to the frequency range ν_s and $\nu_s + d\nu_s$ by h^3, he wrote : *'the total number of cells must be regarded as the number of possible arrangements of a quantum in a given volume'* (our italics). Classically there are infinitely many ways of arranging or distributing a particle within a phase space cell of finite extent corresponding to the infinitely many points contained within it. However, according to Planck, as we have seen, quantum states are not continuously distributed in phase space but are separated from one another by finite amounts determined by h. Thus, *an elementary cell of volume h^3 can contain at most one quantum state*, and the problem of counting the number of quantum states reduces to the counting of these elementary cells. This is a totally new interpretation of the factor $A_s = 8\pi \nu_s^2 V d\nu_s / c^3$. (In classical wave theory it is the number of transverse standing modes in the interval $d\nu_s$ in a volume V.) In his covering letter to Einstein Bose wrote :

> *Respected Sir,*
>
> *I have ventured to send you the accompanying article for your perusal and opinion. I am anxious to know what you think. You will see that I have tried to deduce the coefficient in Planck's law independent of the classical electrodynamics, only assuming that the ultimate elementary regions in Phase-space have the content h^3.*
>
> [Bose to Einstein, 4 June 1924].

On 12 July 1924 Einstein wrote to Ehrenfest : 'The Indian Bose has given a beautiful derivation of Planck's law, including the constant [i.e. $8\pi\nu^2/c^3$].'

Third Step: Having demonstrated that the factor $8\pi\nu^2/c^3$ in Planck's law corresponds to the total number of *quantum states* of radiation, Bose then proceeded 'to calculate *the thermodynamic probability (macroscopically defined) of a state*' in the sense of Planck (our italics). This is equal to the number of different microscopically defined states by means of which the macroscopic state can be realized. If all the cells (states) were considered to be distinct from one another, the answer would be A_s! However, this is not the case. *Since each cell is to be counted as a single quantum state,* if follows that the permutations of the light-quanta within a cell do not produce new states. *Consequently, the cells can be partitioned into distinct classes characterized solely by their occupation numbers*. Let there be a particular distribution of N_s quanta among A_s states such that there are p_1^s states that are empty, p_2^s states with one quantum, p_2^s states with two quanta, and so on. Then, *following Boltzmann* and the usual procedure of statistical mechanics for these *states (not the quanta)*, one can immediately write down the possible number of such distributions as

$$W_s = \frac{A_s \; !}{p_o^s \; ! \; p_1^s \; ! \; p_2^s \; ! \; \ldots} \tag{27}$$

which is precisely the expression (12) given by Natanson. The 'thermodynamic probability' of 'the state defined by all p_r^s' is then clearly $W = \prod_s W_s$ which is equivalent to the Debye-Planck expression (10).

Fourth step : Bose then followed the standard procedure, already followed by Debye [1910], of maximizing *ln W* subject to the constraints $E = \sum_s N_s h\nu_s$ and $N_s = \sum_r rp_r^s$ to obtain Planck's law. Notice that the total number of photons $N = \sum_s N_s$ and *there is no constraint on N*. It turns out that the constraint on N_s is not necessary, but Bose did not notice this.

Notice that the only departure of the expression (27) from the usual Boltzmann expression lies not in the product of factorials in the denominator but in putting the multiplicative factor $(g_i)^r$ corresponding to the possible arrangements of r quanta within each cell i having g_i levels equal to unity, as demanded by the division of phase space into elementary cells of volume h^3. (In his second paper [1924b] written more or less at the same time as the paper under discussion and received by Einstein only five days later, Bose used these factors, as we shall see later, for material particles obeying Maxwell-Boltzmann statistics.) This is precisely the adaptation of statistical mechanics by Planck 'to conform to the requirements of the quantum theory' that Bose referred to in the third paragraph of his paper.

Was Bose aware of the analysis of the Planck-Debye definition of the probability by Natanson and by Ehrenfest and Kamerlingh Onnes? One cannot tell. He certainly did not refer to their papers. It is reasonable, however, to assume that he was not aware of the latter work because if he had been, he would have explained how their conclusion, namely that *'Planck's formal device* (distribution of P energy elements ϵ among N resonators) *cannot be interpreted in the sense of Einstein's light-quanta'* [Ehrenfest and Kamerlingh Onnes, 1914, p.873], could be reconciled with his own starting point, namely the combination of the light-quantum hypothesis with Planck's counting device. The explanation, of course, lies in the fact that Planck's law reduces to Wien's law in the limit $h\nu/kT >> 1$, and that it is only in this limit that the light-quanta have Maxwell-Boltzmann behaviour as inferred by Einstein. It is one of those quirks of history that Einstein should have been able to infer the quantum nature of radiation from its approximate classical statistical behaviour in a certain domain and thus missed the true nature of its statistics. It was left to Bose to discover it, possibly uninhibited by the analysis of Ehrenfest and Kamerlingh Onnes and their conclusion. Hence Einstein's enthusiastic welcome of Bose's work as an important and new support to his favourite light-quantum hypothesis, making it compatible with Planck's law.

j. The birth of a new statistics: Einstein applies Bose's method to ideal gases

Bose's counting of states, of course, implied a *new* statistics for the light-quanta. Neither he nor Einstein seemed to have quite grasped the importance of this initially. In his letter of 2 July 1924 to Bose, Einstein compliments him for being the first to derive the factor A_s quantum theoretically but does not write a word about the new statistics. 'In my opinion,' wrote Einstein in a footnote to Bose's paper, 'Bose's derivation signifies an important advance. The method used here gives the quantum theory of an ideal gas as I will work out elsewhere.' He extended Bose's method to material particles in three communications to the Prussian Academy in Berlin on 10 July 1924, 8 January 1925 and 29 January 1925 [Einstein, 1924, 1925a, 1925b] without informing Bose or asking for his collaboration. Bose came to know of the comment in the footnote shortly before he left for Europe and of the first of the communications to the Prussian Academy on his arrival in Europe in October 1924. In the first paper this is how Einstein described Bose's method :

> [First] : The phase space of an elementary object (here of a monatomic molecule and in Bose's case of a light-quantum), associated with a given (three-dimensional) volume, is divided into "cells" of extension h^3. [Second] : If many elementary objects are present then their (microscopic) distribution, which plays a role in thermodynamics, is determined by the particular manner how the elementary objects are distributed among the cells. [Third] : The "probability" of a macroscopically defined state (in the sense of Planck) is equal to the number of different microscopic states by means of which the macroscopic state can be thought to be realized. [Fourth] : The entropy of the macroscopic state and, therefore, the statistical and thermodynamic behaviour of the system is determined by Boltzmann's theorem relating entropy and probability.' [Einstein, 1924, p.261]

Again, there is no mention of the new statistics! In applying Bose's method to ordinary material atoms, Einstein had to introduce an additional parameter (the chemical potential which vanishes for light-quanta) to take into account the conservation of their number. He drew a very important conclusion from this:

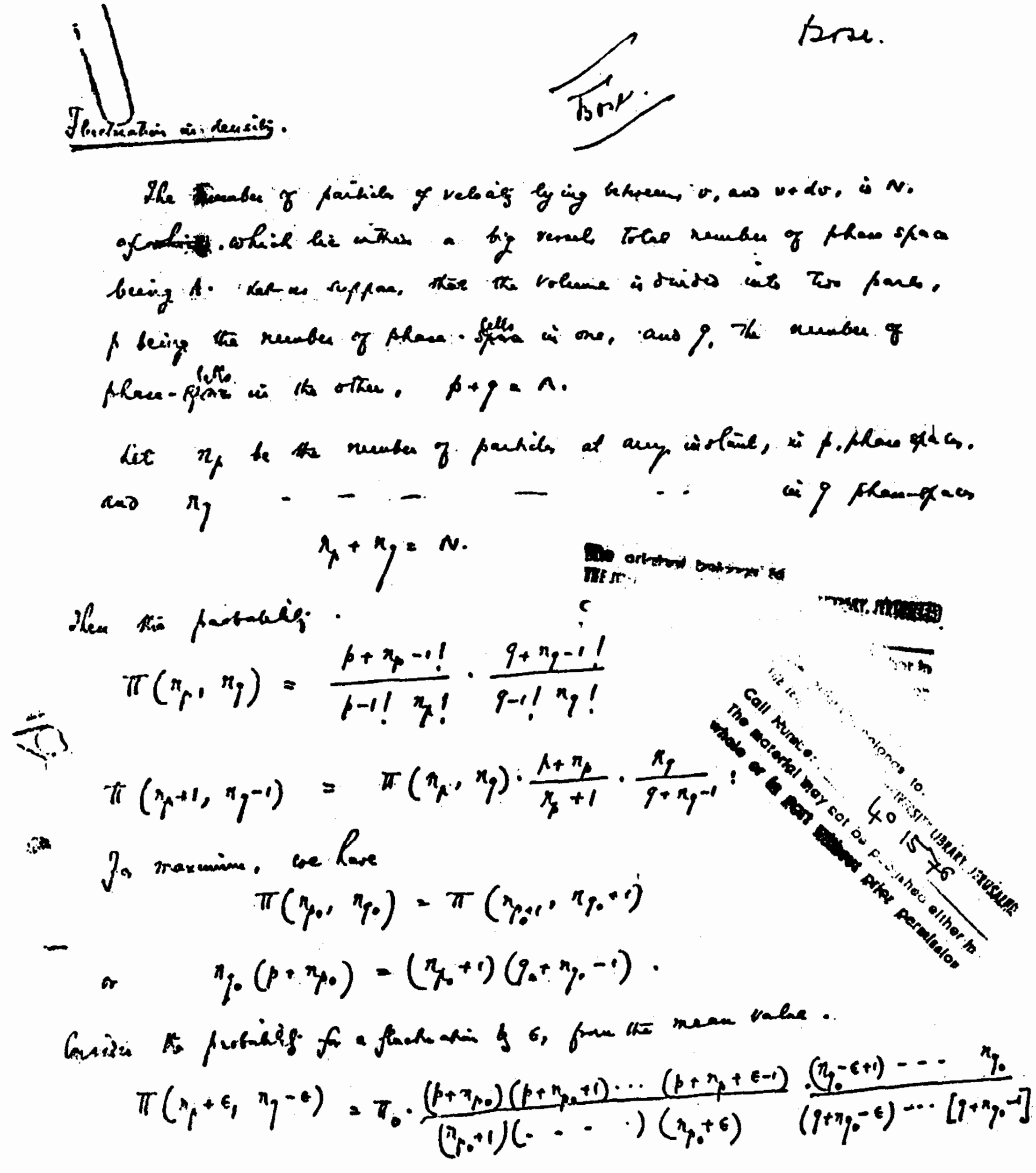
> According to the theory presented here, Nernst's theorem is satisfied in the case of ideal gases. To be sure, our formulae cannot immediately be applied to extremely low temperatures, for we have assumed in their derivation that the p_r^s change only relatively infinitely little if s is altered by 1. Still one recognizes at once that the entropy must vanish at the absolute zero of temperature. The reason is that then *all molecules are in the first cell; and for this state there exists only one distribution of molecules according to our counting method.* Hence our assertion is immediately proved to be correct. [Einstein, 1924, p.265] (our italics)

This is a conclusion of far-reaching consequence, yet there is no mention of any new statistics for the molecules; the emphasis is on Bose's counting method for *states* (one quantum state per cell). It was only six months later on 8 January 1925 that Einstein discussed the difference between the counting methods of Boltzmann and Bose and recognized the fact, already pointed out by Ehrenfest [1924], that *the new statistics for molecules was inconsistent with their statistical independence*. However, while Planck [1925] considered the analogy with radiation a weak point in Einstein's gas theory, the statistics of radiation and that of material particles being completely different in his opinion, Einstein regarded this result differently. He concluded : 'This result represents in itself a support of the view concerning the *deep natural relation between radiation and gas, since the same statistical treatment which leads to Planck's formula establishes — when applied to ideal gases — the agreement with Nernst's theorem.*' [Einstein, 1925a, p.7] (our italics) He was of the opinion that an understanding of the 'mysterious influence' among the molecules would come in the future.

k. The birth of wave mechanics

Based on the new counting method, he showed in his second paper communicated on 8 January 1925 that the mean square energy fluctuation of the gas molecules is given by an expression which is the sum of two terms, one corresponding to the Maxwell-Boltzmann statistics of non-interacting molecules and the other to interference fluctuations associated with wave phenomena. (There exists in the Einstein Archives in the Jewish National and University Library, Jerusalem, a two page calculation done by Bose entitled 'Fluctuation in density' which ends with this fundamental result. The document is undated and exists in the Scientific Correspondence File Folder 'B-Misc.-II'. Bose was in Paris from 18 October 1924 until the first part of October 1925.) From this he immediately drew attention to the importance of Louis de Broglie's Doctoral thesis (which he had heard of from Paul Langevin and asked for a copy which he received and read in December 1924 [Jammer, 1966, 249]) in which de Broglie had attached wave properties to ponderable matter in analogy with the wave-particle duality of radiation. Einstein wrote : 'I shall discuss this interpretation in greater detail because I believe that it involves more than a mere analogy.' He then went on to explore various consequences of de Broglie's hypothesis.

When Erwin Schrödinger whose principal research during his early years in Zurich (1921-27) was on the statistical thermodynamics of ideal gases, read Einstein's first paper [1924a] on the quantum theory of an ideal gas, he also did not understand that the molecules were being treated as being indistinguishable. On 5 February 1925 he wrote to Einstein suggesting a possible error in his probability formula. Einstein replied explaining that the 'quanta or molecules are not treated as independent of one another' and even gave a little diagram to explain the difference between Bose's counting method and that of Boltzmann. He concluded his letter with the emphatic statement: 'There is certainly no error in my calculation.' [Walter Moore, 1989, 183] It was Einstein's remark in his second paper that de Broglie's idea 'involves more than a mere analogy' that really induced Schrödinger to study the 'de Broglie-Einstein undulatory theory' as he called it at that time, and eventually led to his discovery of wave mechanics in 1926 [Schrödinger, 1926a, 1926b and 1926c]. Schrödinger later said that



Bose.

Bose.

Fluctuation in density.

The number of particles of velocity lying between v, and v+dv, is N. of which lie within a big vessel total number of phase space being A. Let us suppose that the volume is divided into two parts, p being the number of phase-cells in one, and q the number of phase-cells in the other, p+q = A.

Let n_p be the number of particles at any instant, in p. phase cells.
and n_q - - - - - - in q phase-spaces

$$n_p + n_q = N.$$

Then the probability.

$$\Pi(n_p, n_q) = \frac{p+n_p-1!}{p-1!\;n_p!}\cdot\frac{q+n_q-1!}{q-1!\;n_q!}$$

$$\Pi(n_p+1, n_q-1) = \Pi(n_p, n_q)\cdot\frac{p+n_p}{n_p+1}\cdot\frac{n_q}{q+n_q-1}$$

For maximum, we have

$$\Pi(n_{p_0}, n_{q_0}) = \Pi(n_{p_0+1}, n_{q_0-1})$$

or $$n_{q_0}(p+n_{p_0}) = (n_{p_0}+1)(q_0+n_{q_0}-1).$$

Consider the probability for a fluctuation of ε, from the mean value.

$$\Pi(n_p+\epsilon, n_q-\epsilon) = \Pi_0\cdot\frac{(p+n_{p_0})(p+n_{p_0}+1)\cdots(p+n_p+\epsilon-1)}{(n_{p_0}+1)(\cdot\;\cdot\;\cdot\;\cdot)(n_{p_0}+\epsilon)}\cdot\frac{(n_{q_0}-\epsilon+1)\;-\;-\;-\;n_{q_0}}{(q+n_{q_0}-\epsilon)\cdots[q+n_{q_0}-1]}$$

Calculation of 'Fluctuation in density' in Bose's hand (Scientific Correspondence File Folder 'B-Misc.-II'); by courtesy of The Jewish National and University Library, Jerusalem.

$$\log \pi (n_{p_0}+\epsilon, n_{q_0}-\epsilon) - \log \pi_0$$

$$= \frac{(\epsilon-1)(\epsilon)}{2}\left[\cancel{\mathrm{bn}}\,\frac{1}{p+n_{p_0}} - \frac{1}{n_{p_0}+1}\right] + \frac{\epsilon(\epsilon-1)}{2}\left[\frac{1}{q+n_{q_0}-1} - \frac{1}{n_{q_0}}\right]$$

or $$\pi(n_{p_0}+\epsilon, n_{q_0}-\epsilon) = \pi_0 \cdot e^{-\frac{\epsilon^2}{2}\left[\frac{p-1}{(p+n_{p_0})(n_{p_0}+1)} + \frac{q-1}{n_{q_0}(q+n_{q_0}-1)}\right]}$$

$$\therefore \overline{\Delta\epsilon^2} = \frac{1}{\frac{p-1}{(p+n_{p_0})(n_{p_0}+1)} + \frac{q-1}{n_{q_0}(q+n_{q_0}-1)}}.$$

where, the volume containing p phase cells is supposed to be very small compared with the other. $\frac{q-1}{n_{q_0}(q+n_{q_0}-1)}$ can be neglected.

$$\therefore \frac{\overline{\Delta\epsilon^2}}{n_{p_0}^2} = \frac{(p+n_{p_0})\,n_{p_0}}{p\,n_{p_0}^2} = \frac{p+n_{p_0}}{p\,n_{p_0}} = \left(\frac{1}{p}+\frac{1}{n_{p_0}}\right)$$

'*wave mechanics was born in statistics.*' (our italics) [Walter Moore, 1989, 188] He also wrote to Einstein on 23 April 1926 : 'By the way, the whole thing would not have started at present or at any other time (I mean, as far as I am concerned) had not your second paper on the degenerate gas directed my attention to the importance of de Broglie's ideas.' [Jammer,1966, 257]

l. Further developments

Meanwhile in July 1925 Werner Heisenberg had sent a paper to *Zeitschrift für Physik* in which he gave a preliminary account of matrix mechanics [W Heisenberg,1925]. Schrödinger proved the complete equivalence of his wave mechanics and the Heisenberg–Born–Jordan matrix mechanics [Schrödinger, 1926d]. A whole train of rapid developments occurred. Pauli discovered his Exclusion Principle within six months of Bose's paper [Pauli, 1925]; Fermi published his paper on the statistics of particles obeying Pauli's principle in early 1926 [Fermi, 1926]; Dirac linked the Bose and Fermi statistics of particles to the symmetry properties of their wave functions and named them 'bosons' and 'fermions' [Dirac, 1926]; in 1927 Dirac invented 'second quantization' of the electromagnetic field [Dirac, 1927] which was soon followed by a similar method for Fermi fields developed by Jordan and Wigner [1928]; later Pauli [1940] proved the spin-statistics theorem in relativistic quantum field theory; Pauli [1927] also applied Fermi-Dirac statistics to the paramagnetism of an electron gas and Sommerfeld [1927,1928] applied it to the electrons in a metal; and Fritz London [1938] suggested that the superfluidity of liquid helium (^{4}He) was related to Einstein-Bose condensation. Thus the developments of Schrödinger's wave mechanics and that of quantum statistical mechanics were directly triggered off by Bose's first paper followed by Einstein's far-sighted extension of his method to ideal gases of *material* particles and his championing of de Broglie's ideas which soon received experimental confirmation in the hands of Davisson and Germer [1927] and G P Thomson [1927].

Years later this is how Schrödinger summed up the position :

> The two equivalent ways of looking at (7.19) either as counting the number of quantum states of a particle, or as counting the number of wave-mechanical proper vibrations of the enclosure, interest us for this reason. The second attitude makes us think of the 'n_s particles present in state α_s' as of a proper vibration (or a 'hohlraum' oscillator to use a customary phrase) in its n_sth quantum level. (This attitude really corresponds to so-called second quantization or field quantization.) n_s becomes a quantum number and the stipulation that the system of quantum numbers
>
> $$n_1, n_2, n_3, \ldots, n_s, \ldots,$$
>
> determines only one state of the gas, not a class of
>
> $$\frac{n\,!}{n_1\,!\; n_2\,! \ldots .\; n_s\,! \ldots}$$
>
> states, ceases to be a strange new adoption, and comes into line with the ordinary view about quantum states and their statistical weight (viz. equal

> for any two of them).
>
> It is the first, the particle attitude, that has suggested the term 'new statistics' which is frequently used. And that is why this idea of new statistics did not, originally, arise in connexion with heat radiation, because here the wave point of view was the historical one, the classical one—nobody thought of any other at the outset. The wave picture was considered to be (and historically was) the classical description. The quantization of the waves therefore duly appeared to be a 'first' quantization and nobody thought of anything like 'second quantization'.
>
> Not until the idea of photons had gained considerable ground did Bose (about 1924) point out that we could, alternatively to the 'hohlraum' oscillator statistics, speak of photon statistics, but then we had to make it 'Bose statistics'. Very soon after, Einstein applied the same to the particles of an ideal gas. And thereupon I pointed out that we could also in this case speak of ordinary statistics, applied to the wave mechanical proper vibrations which correspond to the motion of the particles of the gas [Schrödinger, 1952, 49-50].

We wish only to point out, as explained above, that Bose did not actually 'speak of photon statistics' — it was *implied* by his counting of quantized photon *states*, and this only became gradually clear with Einstein's application of Bose's counting method to ideal gases, although Ehrenfest and Natanson had already pointed this out in 1911 and 1914 respectively in relation to Planck's counting method of December 1900 which Bose showed was applicable to light-quanta.

m. Did Bose make a 'shot in the dark'?

Some recent widely read scientific biographies have given rise to the impression that Bose made 'a shot in the dark' with three assumptions all of which turned out to be correct but none of which he tried to justify 'by arguments of any kind; they seemed to appear intuitively to his mind, perhaps because his thoughts were uncluttered by the ongoing controversies of the European physicists.' [W. Moore, 1989, 181; A. Pais, 1982, 428] The assumptions were that photons were (a) massless particles capable of existing in two states of polarization, (b) indistinguishable and (c) their number was not necessarily conserved. The record must be put right. That photons are massless particles with energy $h\nu$ and directed momentum $h\nu/c$ was proposed by Einstein and only used by Bose. That they exist in two possible 'polarization states' was *inferred* by Bose from phase space considerations which yield for the number of quantum states between ν_s and $\nu_s + d\nu_s$ the factor $4\pi V\nu_s^2\, d\nu_s/c^3$ rather than $8\pi V\nu_s^2\, d\nu_s/c^3$, the factor that occurs in Planck's formula. Actually, as we have already discussed, in order to account for this factor of 2, Bose had proposed that photons carried one unit of intrinsic spin which could only take the values ±1. Whoever had heard of particles with polarization? Moreover, 'polarization' was a wave concept which Bose wanted to avoid. However, Einstein quietly dropped it and the statement regarding the polarization factor was most probably inserted by him. Since Bose's original manuscript in English is missing from the Einstein archives and Bose did not keep a copy, there is no way to verify this.

How about 'indistinguishability'? Again, as we have seen, although Bose did not explicitly use the word 'indistinguishable' which is by itself rather vague, he did use Planck's division of phase space into elementary cells of volume h^3 to justify his counting method for *states* (which implied indistinguishability of the quanta), namely that there cannot be more than one quantum state in an elementary cell. 'In any case,' he wrote, 'the total number of cells must be regarded as the number of possible arrangements of a quantum in the given volume.' What is not usually appreciated in usual expositions of Bose statistics is that Bose did not count the number of ways of distributing individual photons over a set of states. Instead, he calculated the *thermodynamic probability* of a *macroscopic state* by counting the number of ways in which the macroscopic state can be thought to be realized through given distributions of the number of photons over the micro-states (phase space cells).

Finally, what about non-conservation of photons? This is a non-issue. Since, like Debye [1910], Bose considered the statistical thermodynamics of an *open* system of radiation in thermal equilibrium and did not include the molecules of matter with which this radiation constantly exchanged energy (which he did in his second paper which we shall turn to presently), there was no question of conservation of energy quanta even if treated as massless particles. In his 1917 derivation of Planck's law Einstein did not impose the conservation of light-quanta which he certainly regarded as massless particles interacting with molecules. Since Bose was not doing anything new or different in this respect, there was no original or deep comment he could have made on this issue. Nevertheless, it is true that Bose, like most others who have made revolutionary contributions, did not, and could not be expected to, understand the full implications of their discoveries.

n. Bose's second paper (1924)

We now turn to his second paper [1924] on *'Thermal Equilibrium in Radiation Field in the Presence of Matter'*, also translated into German probably by Einstein himself. This paper, completed on 14 June 1924, i.e. more or less at the same time as the first paper, consists of two parts. In the first part Bose derived general conditions for statistical equilibrium of a system consisting of matter and radiation, *independent of any special assumptions about the mechanism of the elementary radiative processes.* In the second part he proposed a new expression for the probability of these elementary radiative processes which differed from that of Einstein. Einstein considered this hypothesis not to be 'applicable' to elementary radiative processes and gave two reasons for it in a footnote to the paper. As a consequence, the paper has been largely ignored. However, Bose remained firmly of the opinion for the rest of his life that Einstein had not done him justice.

Bose started off by giving a quick critical review of the most important derivations of Planck's law until then given by Debye [1910], Einstein [1917], Pauli [1923], and Einstein and Ehrenfest [1923] on the scattering of radiation by electrons. Bose considered Debye's 1910 derivation not to be 'completely independent of classical electrodynamics' since he derived the factor $8\pi\nu_s^2 \, V d\nu_s/c^3$ from the normal modes of the ether.

He considered the derivations of Einstein [1917], Pauli [1923] and Einstein and Ehrenfest [1923] to be dependent on *ad hoc* special assumptions about the probability of elementary radiative processes. 'The problem of thermodynamic equilibrium of radiation in the presence of material particles,' he wrote, 'can however be studied using the methods of statistical mechanics, independently of any special assumption about the mechanism of the elementary processes on which the energy exchange depends. In this way we obtain a general relation which is valid for all special assumptions about the elementary processes and their probabilities.' This is in keeping with Kirchhoff's law. He then went on to say: 'If it is possible to write down the thermodynamic probability for any particular distribution of quanta of radiation and the probability for any arbitrary energy distribution among the particles, then the thermodynamic probability for the bigger system consisting of matter and radiation is simply the product of the two probabilities. The condition of equilibrium is simply that the resulting probability must be a maximum.' For the thermodynamic probability for radiation, Bose used the expression

$$W = \prod_{\nu} \frac{(A_\nu + N_\nu \, d\nu)\,!}{A_\nu\,!\, N_\nu \, d\nu\,!} \tag{28}$$

where $A_\nu = 8\pi V\nu^2 \, d\nu/c^3$ which, he stated, had been derived earlier. As references he quoted his own paper and that of Debye [Debye 1910]. (It is curious that the reference to his previous paper is given as 'The author, to appear in *Phil Mag*'. This is because Bose had submitted his paper first to the *Philosophical Magazine*. Not hearing from them for a while, he sent a copy to Einstein. While translating Bose's second paper, the translator obviously forgot to change the reference of the earlier paper to *Zeitschrift für Physik* **26**, 178-181, 1924). Although expression (28) is indeed the Ansatz used by Debye, it does not appear in Bose's first paper! Instead, the corresponding expression that appears in his first paper is (27), the form given by Natanson. As we have seen earlier, the two are mathematically equivalent, but Bose did not care to explain this, probably because this result must have been fairly well known in the literature in those days.

For the thermodynamic probability for material particles. Bose writes:

> This can be found easily. We want to generalize our assumptions a little so that the case of the Bohr atom with discrete energy levels as well as the case of translational energy of particles can be included. *Let the phase space be divided into cells. For every cell there is a probability g that a particle occupies it. The g's are in general equal except for the case of Bohr's atoms. The thermodynamic probability for any arbitrary distribution* n_1, n_2 etc among the different cells is,

$$\frac{N\,!\; g_1^{n_1}\, g_2^{n_2} \ldots}{n_1\,!\; n_2\,! \ldots} \tag{29}$$

(our italics). Unlike in the case of radiation (expression (27), Bose now uses the factor $\prod_i g_i^{n_i}$ for material particles and obtains the classical Maxwell-Boltzmann distribution. In other words, he treats material particles as in classical (*distinguishable*) statistical mechanics but light-quanta according to quantum (*indistinguishable*) statistical mechanics.

Finally, he writes down the thermodynamic probability for the total system as the product of (28) and (29) :

$$W = \prod_s \frac{(A_s + N_s)\,!}{A_s\,!\; N_s\,!} \prod_t \frac{g_T^{n_T} N\,!}{n_T\,!} \tag{30}$$

with

$$\Sigma N_s h\nu_s + \Sigma n_T E_T = E \tag{31}$$

and

$$\Sigma N_T = N. \tag{32}$$

He then considers an elementary process in which a particle passes from the *r*th cell to the *s*th cell while a light-quantum of frequency ν changes into a light-quantum of frequency ν.′ The stationarity of W subject to the conditions (31) and (32) gives

$$\frac{n_r}{g_r} \prod_\nu \frac{N_\nu}{N_\nu + A_\nu} = \frac{n_s}{g_s} \prod_{\nu'} \frac{N_{\nu'}}{N_{\nu'} + A_{\nu'}} \tag{33}$$

where

$$\Sigma\, h\,\nu' - \Sigma\, h\nu + E_s - E_r = 0\,. \tag{34}$$

Bose then demonstrated that (33) indeed generalized the results of Pauli [1923] and Einstein and Ehrenfest [1923] without the need for any *ad hoc* assumptions regarding the elementary radiative processes designed to give Planck's formula. In particular, in the case of Bohr's atoms considered by Einstein in 1917, equation (33) reduces to

$$\frac{n_r}{g_r} \frac{N_\nu}{N_\nu + A_\nu} = \frac{n_s}{g_s} \tag{35}$$

which is Einstein's condition derived on the assumption that atoms in higher energy levels make transitions to lower energy levels in two ways:

1. A spontaneous transition as in radioactivity *independent of the state of the external radiation field.*
2. An induced transition whose probability depends on the state of the external radiation field.

Transitions from lower to higher energy levels always take place by induced absorption whose probability depends on the state of the external radiation field. Further, Einstein had to postulate certain relations (namely, the opposite limiting Wien and Rayleigh-Jeans laws) among these transition probabilities to obtain Planck's law. Bose's derivation is independent of all these assumptions. It is for this reason that Einstein took the trouble to have the paper translated into German and communicated it to *Zeitschrift für Physik*. However, his disagreement with Bose started from the *next* part of the paper in which Bose proposed a new expression for the probability of an interaction between a particle and a quantum of radiation.

Bose's starting point was the following observation which he considered to be fundamental :

> . . . *even in a collision no interaction is as probable as the occurrence of any special interaction. . . From the classical theory. . . one would expect that there is some interaction whenever a quantum and a material particle come together. Therefore here it is a question of a departure from classical theory.*
>
> The assumption made above, that even in a collision no interaction need occur, is quite analogous to the assumption of the stability of stationary states which is so fundamental to Bohr's theory of line emission and can be traced back to the same origin—*the probability of existence of the stationary states of the particles*. It is interesting to remark in this connection that in the analogous case of a collision of an electron with an atom, experiments show that the electron goes through the atom without changing either the interatomic or its own motion. (our italics)

Bose was obviously referring to the Ramsauer–Townsend effect. He saw a simple way of realizing the non-classical possibility in Planck's cellularization of phase space. Let p_r be the number of cells with r quanta. 'In order that the radiation and particle may interact, it must be in a cell occupied by a quantum. The particular cell which the molecule under consideration occupies will pass through all possible states if we observe it long enough, that is, sometimees it is empty, sometimes occupied by 1 quantum, sometimes by 2 quanta etc. The duration of these events will be finally proportional to the equilibrium values of p_o, p_1 etc.*When r quanta and a particle are together, then either an exchange of energy takes place or nothing happens.*' (our italics) Therefore $r + 1$ different events are possible, namely the exchange of 1 quantum of energy, 2 quanta etc or no exchange at all. Therefore the total number of possible events is

$$p_o + 2p_1 + 3p_2 + \ldots = A_\nu + N_\nu d\nu_s = \sum_r (r+1) p_r . \qquad (36)$$

The number of cases in which interaction or energy exchange occurs is

$$p_1 + 2p_1 + 3p_3 + \ldots. = N_s d\nu_s = \sum_r r p_r . \qquad (37)$$

Consequently the probability of an interaction is

$$P = \frac{\Sigma\, r\, p_r}{\Sigma\,(r+1)\,p_r} = \frac{N_s\, d\nu_s}{A_s + N_s\, d\nu_s}\,. \tag{38}$$

This is Bose's second fundamental result.

Having derived this result, Bose then considered Einstein's 1917 problem, namely Bohr atoms exchanging energy with the radiation field. In order to be consistent with the fundamental equilibrium condition (35), he found it was sufficient to consider (a) transitions from lower to higher states through absorption of quanta of characteristic frequency ν and (b) transitions from higher to lower states to be spontaneous, i.e. independent of the external radiation field. *The additional assumption of induced emission (negative radiation) processes was not necessary*. Let the strength of induced absorption be β. Then the probabillity of induced absorption is

$$\beta\,\frac{N_\nu\, d\,\nu}{A_\nu + N_\nu\, d\nu}\,. \tag{39}$$

Let the coefficient of spontaneous emission be α. Then for equilibrium one gets

$$n_r\,\beta\,\frac{N_\nu\, d\nu}{A_\nu + N_\nu\, d\nu} = \alpha\, n_s \tag{40}$$

which agrees with (35) provided $g_r\,\beta = g_s\,\alpha$. A substitution of the value of n_r/n_s then gives Planck's law. Bose then treated the cases considered by Pauli [1923] and Einstein and Ehrenfest [1923] and showed that the fundamental equilibrium condition (33) followed from the probability law (38) without requiring any further assumptions.

In a note added to the paper Einstein gave two reasons why he thought Bose's hypothesis about the probability of elementary radiative processes was not 'applicable'. In his letter of 3 November 1924 Einstein summarized the two reasons. 'Your principle is not compatible with the following two conditions:

1) The absorption coefficient is independent of the radiation density.
2) The behaviour of a resonator in a radiation field should follow from the statistical laws as a limiting case.' [Einstein to Bose, 3 November 1924]

Let us see to what extent these criticisms were fair. As far as the first point is concerned Einstein was referring to Beer's law. Bose's probability law, however, predicted a dependence of the absorption coefficient on the radiation density , decreasing with it. 'If the behaviour had been such,' wrote Einstein in the note, 'then it would certainly have already been discovered in the case of infra-red radiation from hot light sources.' This is hard to believe. The departure from classical behaviour predicted by Bose's principle should occur only when $N_\nu / A_\nu << 1$, i.e. for very low intensity radiation. It is now well known that the departures from classical behaviour predicted by quantum optics are not seen even when the intensity of light is extremely low, *unless the source happens to be of a special kind* (like a single-photon source). The reason is that light from all classical sources is in a coherent state in which the number fluctuation is extremely large (Poissonian or higher). Therefore only light from 'single-photon states' (Fock states) and 'squeezed states' and other such 'sub-Poissonian situations' show departures from classical behaviour. [Aspect, 1987; Loudon, 1983] Such light sources were certainly unknown in 1924.

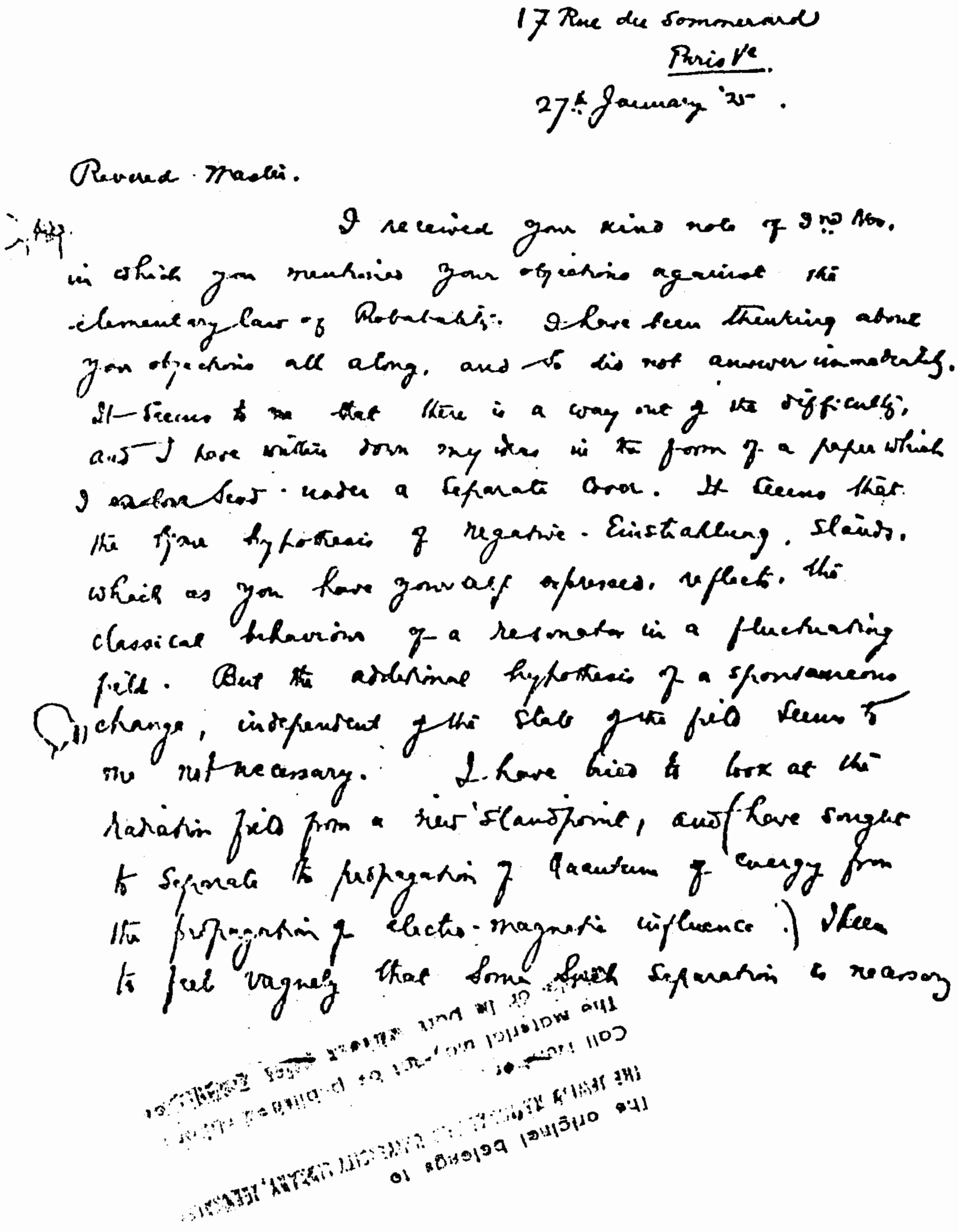
The second point concerns the 'correspondence principle' (not necessarily Bohr's) which in this case implies that a radiation field can transfer positive as well as negative energy to a resonator, depending on the phase. 'The probabilities of both these transitions must depend on the density of radiation, that is, on N_ν as opposed to Bose's hypothesis,' wrote Einstein. The discovery of masers and lasers after the Second World War eventually vindicated Einstein's contention. It was nevertheless eventually an *empirical* rather than a sound theoretical refutation. Although the 'correspondence principle' was used as a heuristic tool with great success by Einstein, Planck and later the Copenhagen school prior to the discovery of quantum mechanics around 1925/1926, it is no longer regarded as a fundamental principle. In fact, quantum mechanics itself is known not to satisfy this principle. [Liboff, 1980; Cabrera and Kiwi, 1987; Bohm and Hiley, 1985] For example, the nodes of the eigen-solutions inside a box do not disappear for arbitrarily high quantum numbers so that a classical particle cannot pass through such points. Also the cross-section for hard sphere scattering has no classical limit. Einstein was aware of this and used it against quantum mechanics as well.

o. Bose's third (unpublished) paper (1925)

Although Bose was unhappy, particularly with the first point ragarding Beer's law, he accepted the second point, modified his stance and wrote a third paper which he sent to Einstein from Paris on 27 January 1925. He sent him a letter under separate cover in which he wrote :

> *It seems that the hypothesis of negative Einstrahlung stands, which, as you have yourself expressed, reflects the classical behaviour of a resonator in a fluctuating field. But the additional hypothesis of a spontaneous change, independent of the state of the field, seems to me not necessary. . .*
>
> *I am rather anxious to know your opinion about it. I have shown it to Professor Langevin here and he seems to think it interesting and worth publishing.*

It was never published. Neither did Bose keep a copy of the paper, nor is there one in the Einstein Archives as far as one can ascertain. So, the precise contents and details will remain unknown. Bose arrived in Berlin in October 1925 and discussed his new ideas with Einstein. Einstein was so convinced of the necessity and reality of the two independent processes of emission of light from atoms (spontaneous and induced) that he could not agree with Bose that one of them, the former, was unnecessary to assume in deriving Planck's law. To Bose, the process of emission of light appeared as *a single physical process* and the division into two independent processes appeared artificial. The assumption of one, he claimed, would automatically imply the other in thermal equilibrium, *given the special statistical property of the radiation field.* Initially Bose started with spontaneous emission as input but later changed his position. It was clear to Bose that Planck's law was *a consequence of the special statistical property of the radiation field itself and was independent of the mechanisms of energy transfer with*

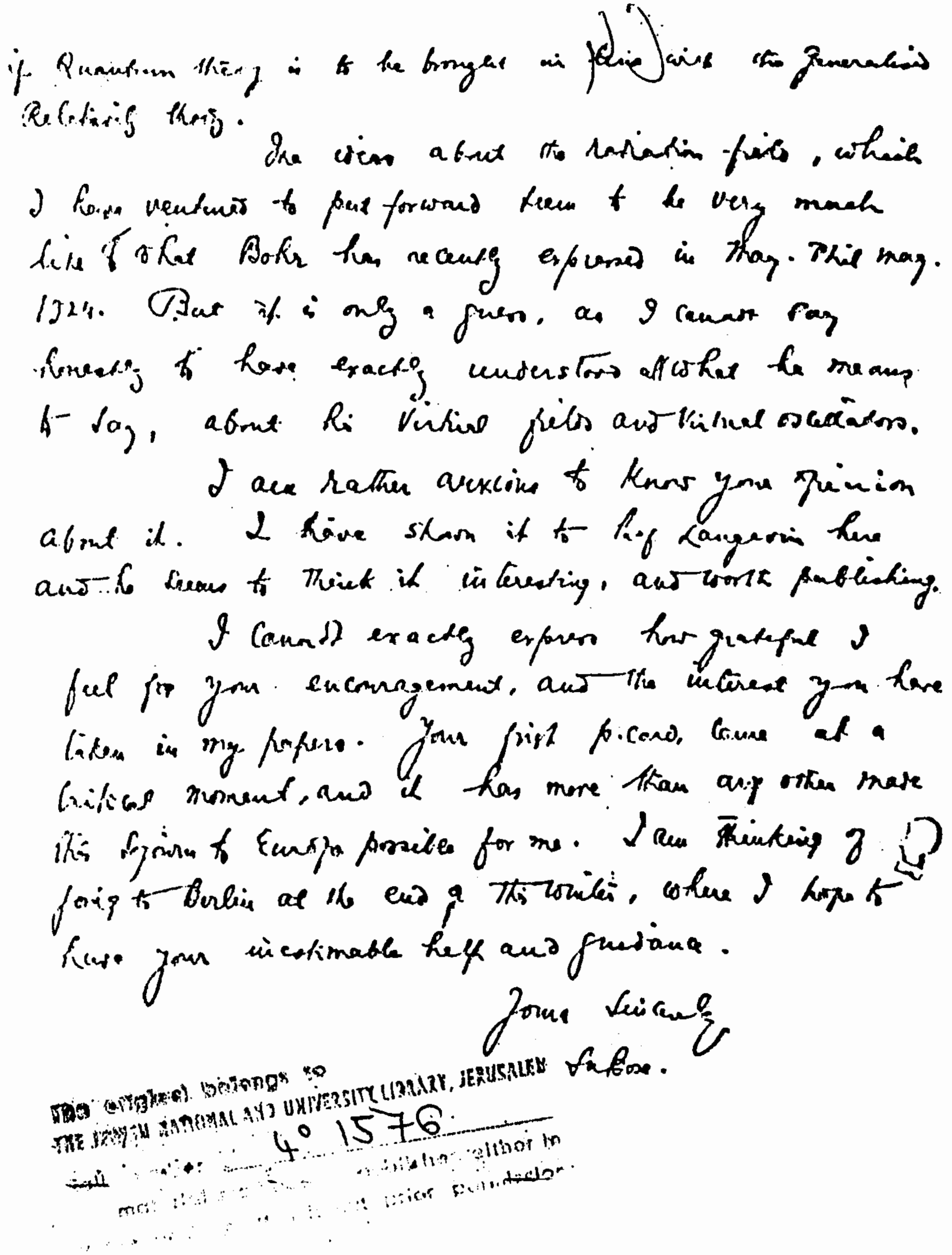


17 Rue du Sommerard
Paris Ve.
27th January '25.

Revered Master.

I received your kind note of 3rd Nov. in which you mention your objections against the elementary law of Probability. I have been thinking about your objections all along, and so did not answer immediately. It seems to me that there is a way out of the difficulty, and I have written down my ideas in the form of a paper which I enclose sent under a separate cover. It seems that the first hypothesis of negative Einstrahlung stands, which as you have yourself expressed, reflects the classical behaviour of a resonator in a fluctuating field. But the additional hypothesis of a spontaneous change, independent of the state of the field seems to me not necessary. I have tried to look at the radiation field from a new standpoint, and (have sought to separate the propagation of quantum of energy from the propagation of electro-magnetic influence.) I seem to feel vaguely that some such separation is necessary

Bose's letter of 27 January 1925 to Einstein



if Quantum theory is to be brought in harmony with the Generalised Relativity theory.

The ideas about the radiation field, which I have ventured to put forward seem to be very much like to what Bohr has recently expressed in Phil. Mag. 1924. But it is only a guess, as I cannot say honestly to have exactly understood all what he means to say, about his virtual fields and virtual oscillators.

I am rather anxious to know your opinion about it. I have shown it to Prof. Langevin here and he seems to think it interesting, and worth publishing.

I cannot exactly express how grateful I feel for your encouragement, and the interest you have taken in my papers. Your first p.card came at a critical moment, and it has more than any other made this sojourn to Europe possible for me. I am thinking of going to Berlin at the end of this winter, where I hope to have your inestimable help and guidance.

Yours sincerely

S.N. Bose.

atoms and molecules. This was indeed his capital contribution to physics and forms the substance of his first paper and the first part of his second paper. To Einstein, on the other hand, it appeared that Planck's law was a consequence of special mechanisms of energy transfer between the radiation field and molecules and that thermal equilibrium was impossible without both induced and spontaneous emission. This was indeed the lesson he derived from his 1916/1917 papers which he considered to contain 'the fundamental outline of the future theoretical derivation.' Einstein was compelled to regard spontaneous emission as being independent of the state of the radiation field and an *inherent* property of atoms and molecules because in 1917 he was unaware of the special quantum statistical property of the radiation field. It is strange that even after championing Bose's method he failed to see Bose's point about spontaneous emission. To Bose it was clear that it depended on the 'environment' in which the atom or molecule was placed. He wrote in his second paper [1924] :

> From classical theory one would expect that no stationary states are possible and that an interaction or coupling (Bohr) between ether and the excited atom and the radiation connected with it is always present. On the other hand, in order to explain spectral emission we must assume it to be possible that no interaction (emission) occurs. Because of the existence of stationary states we are therefore led to assign to every stationary state a probability coefficient or a mean life-time.

Recall the expression (38) for the fundamental probability *P*. It can be written as $\bar{n} / (\bar{n} + 1)$ where $\bar{n} = N_v / A_v$ is the average number of quanta per cell. For $\bar{n} >> 1$, *P* tends to unity, the classical limit in which stationary states are not possible. It follows from this that the average life-time of a stationary state (and therefore the probability of spontaneous emission) depends on the average radiation density $\bar{n}$. Bose used to recount to his students Einstein's objection to this. 'Suppose there were only one hydrogen atom in the universe in an excited state,' Einstein once remarked to Bose, 'don't you think it would radiate spontaneously and come down to its ground state?' Bose did not disagree, but nevertheless, felt that such a *gedanken* situation did not capture the situation in a Hohlraum or cavity. It was Bose's intuition that eventually turned out to be right in the end, not Einstein's. Firstly, in Dirac's method of 'second quantization' (Dirac, 1927) which is widely used in quantum field theory today, one starts with a classical electromagnetic field in which the equality of the probabilities of induced emission and absorption (determined by the Fourier coefficients in the plane wave expansion) is built in, and obtains the spontaneous emission term as a *consequence* of the fundamental commutation rules. Although this is not exactly what Bose had proposed, it fits in with his basic contention that spontaneous emission is a consequence of the quantal property of the radiation field *itself* and need not be introduced as an independent hypothesis concerning the radiating atoms. Einstein's initial reaction to Dirac's contribution was also decidedly negative. [Pais, 1982, 441]

Secondly, in quantum electrodynamics spontaneous emission is not a property of an isolated atom but of an atom-vacuum system and can be significantly inhibited or enhanced by placing the atoms in a suitable environment. In quantum electrodynamics the vacuum is not empty and devoid of everything—it is full of virtual particles appearing and disappearing, interacting with one another; it provides a non-trivial physical 'environment' to an atom. A whole new branch of quantum optics called 'cavity quantum electrodynamics' has developed since about 1987 utilizing dramatic changes in spontaneous emission rates to construct new kinds of microscopic masers that operate with a single atom and a few photons or with photons emitted in pairs in a two-photon transition. This is what two of the pioneers in the field have to say :

> Ever since Einstein demonstrated that spontaneous emission must occur if matter and radiation are to achieve thermal equilibrium, physicists have generally believed that excited atoms inevitably radiate. Spontaneous emission is so fundamental that it is usually regarded as an inherent property of matter. *This view, however, overlooks the fact that spontaneous emission is not a property of an isolated atom but of an atom-vacuum system.* The most distinctive feature of such emission, irreversibility, comes about because an infinity of vacuum states is available to the radiated photon. If these states are modified — for instance, by placing the excited atom between mirrors or in a cavity– spontaneous emission can be greatly inhibited or enhanced.
>
> Recently developed atomic and optical techniques have made it possible to control and manipulate spontaneous emission. [Haroche and Kleppner, 1989]

In his third unpublished paper Bose had gone further. He wrote in his letter of 27 January 1925 to Einstein :

> *I have tried to look at the radiation field from a new standpoint and have sought to separate the propagation of Quantum of energy from the propagation of electro-magnetic influence. I seem to feel vaguely that some such separation is necessary if Quantum theory is to be brought in line with Generalized Relativity theory.*
>
> *The views about the radiation field, which I have ventured to put forward, seem to be very much like what Bohr has recently expressed in May Phil Mag 1924. But it is only a guess, as I cannot say honestly to have exactly understood all he means to say about virtual fields and virtual oscillators.*

This letter shows that Bose was one of the first persons to have formulated the idea of an 'empty wave' (an electromagnetic wave propagating in space and time but carrying no energy-momentum). He was certainly the first person to have intuitively seen its relationship with the principle of General Relativity, although the connection is not clear. Einstein already had the idea of an empty wave in his light-quantum hypothesis according to which 'a ray of light expands starting from a point, the energy does not distribute on ever increasing volumes, but remains constituted of a finite number of energy quanta localized in space and moving without subdividing, and unable to be absorbed or emitted partially.' [Einstein, 1905]

If the localized quanta carry all the energy and momentum, what happens to the electromagnetic waves which can produce interference? There was no clear-cut answer. The problem was so acute that Einstein referred to these waves as *Gespensterfelder* (ghost waves) guiding the photons. [Bohr, 1945, 206] Louis de Broglie also had a similar idea in his concept of phase or pilot waves 'guiding the propagation of the energy' and enabling a 'synthesis of the waves and the quanta.' [de Broglie, 1923, 549] Slater also arrived at a similar notion. As he recalled : 'A number of scientists — W [illiam] F [rancis] G [ray] Swann, among others — had suggested that the purpose of the electric field was not to carry a continuously distributed density of energy, but to guide the photons in some manner. This was the point of view which appealed to me, and during my visit at the Cavendish Laboratory in the fall of 1923, I elaborated on it. [Slater, 1975, 9] Slater who believed in the existence of light-quanta as well as electromagnetic waves found the following way out of the difficulty : 'I decided you could only have a statistical connection between them because you couldn't set up a vector to represent the photon in electromagnetic theory : that you would have the intensity of the wave governing the probability of finding the photon there. And I wanted to have the wave emitted during the stationary state so as to get it emitted over a long enough period so that it would have a suitable spectral distribution.' [Slater, loc.cit., 30]

When Slater arrived in Copenhagen on 21 December 1923 to collaborate with Hendrik Kramers and Niels Bohr, he was prevailed upon by them to relinquish Einstein's light-quantum hypothesis but to retain his idea of electromagnetic waves emitted by oscillators during their stationary states and carrying no energy. They coined the name *virtual oscillators* for them. The collaboration resulted in the famous Bohr–Kramers–Slater proposal [Bohr, Kramers, Slater, 1924] which Bose referred to in his letter to Einstein. In the BKS proposal the conservation of energy (and momentum) was abandoned for individual transition processes in atoms but was retained as a statistical concept. Most interestingly, they wrote : 'the transitions which in [the Einstein theory of 1917] are designated as spontaneous are, in our view, induced by the virtual field.' Bose must have noted this with satisfaction and this must have been one reason why he regarded the BKS proposal to be 'very much like' his own. The theory was however quickly falsified by experiments carried out by Bothe and Geiger [Bothe and Geiger, 1925] and Compton and Simon [Compton and Simon, 1925a, b] which established the conservation of energy and momentum in individual processes. Nevertheless, their ideas were reformulated and generalized by Born [1926] in his well known statistical interpretation of the wave function in quantum mechanics, an idea now universally accepted. This is what Heisenberg wrote about its historical antecedent :

'The probability wave of Bohr, Kramers and Slater. . . was a quantitative version of the old concept "potentia" in Aristotelian philosophy. It introduced something standing in the middle between the idea of an event and the actual event, a strange kind of physical reality just in the middle between possibility and reality.

Later, when the mathematical framework of quantum theory was fixed, Born took up this idea of the probability wave and gave a clear definition of the mathematical quantity in the formalism.' [Heisenberg, 1958, 29]

On 16 July 1946 , twenty-one years after he received Bose's letter of 27 January 1925 from Paris, Einstein wrote to Schrödinger about the latest status of the unified field theory (of electromagnetism and gravitation based on the General Principle of Relativity):

'It is indeed correct that the energetics occurs quite differently here compared to Maxwell-Poynting. I consider, however, that in this respect the Maxwell theory is really false on account of quantum actualities. In light the energy exists in something like quasi-singularities. *The wave field as such should therefore not be the site of energy. One must simply require this and be happy that in the new theory the transverse wave field is indeed present but as such transports no energy.'* (Our italics) [Moore, 1989, 427]

There is no mention of Bose, his letter or his third paper which Einstein must have rejected outright, although the facsimile of Bose's letter supplied by The Jewish National and University Library, Jerusalem, shows that he had marked the phrase 'have sought to separate the propagation of Quantum of energy from the propagation of electro-magnetic influence' within parentheses.

Louis de Broglie's original interpretation of Schrödinger's equation [de Broglie, 1927] was given a firmer basis by Bohm [1952]. In the de Broglie-Bohm 'casual interpretation' particles are thought to exist independent of observation, embedded in a wave field ψ which satisfies the Schrödinger equation, with a density (over the statistical ensemble) of value $|\psi|^2$. One can show that this approach can explain the interference pattern observed in double-slit experiments without abandoning the notion of well-defined particle trajectories. [Philippidis, Dewdney and Hiley, 1979] Experiments for testing certain consequences of 'empty waves' have been proposed [Selleri, 1990, ch. 4] and some have already been carried out [Zou, Grayson, Wang and Mandel, 1992]. However, such experiments have not yet ruled out all versions of 'empty wave' theories [Vigier and Holland, 1991].

Bose remained ever grateful to Einstein (whom he always addressed as 'Sir' or 'Master') for his encouragement and the interest he had taken in his first paper. Nevertheless, deep within himself, he felt a resentment that rankled all through his life. Three of his most perceptive and original ideas were scotched by Einstein :

1. The idea of the photon spin which is now established beyond doubt. (That Bose proposed the idea in his first paper in 1924 cannot as yet be established beyond every shred of doubt, although circumstantial evidences are strong.)
2. The idea that spontaneous emission is not an inherent property of an isolated atom, totally independent of the radiation field, also confirmed by experiments since 1987.
3. The idea of the propagation of electromagnetic influence carrying no quanta of energy, an area of current theoretical and experimental activity.

Bose could not publish a single paper during his two years in Europe at a time when physics was going through a turmoil and one of its most exciting periods. He returned to Dhaka a disheartened man and did not publish anything in theoretical physics for the next twelve years !

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S N Bose : Collected scientific papers

On the Influence of the Finite Volume of Molecules on the Equation of State.[†] *By* MEGH NAD SAHA, *M. Sc., and* SATYENDRA NATH BASU, *M. Sc., Lecturers on Mathematical Physics, Calcutta University.*

It is wellknown that the departure of the actual behaviour of gases from the ideal state defined by the equation $p = \frac{NK\theta}{v}$ is due to two causes : (1) the finiteness of the volume of the molecules, (2) the influence of the forces of cohesion, i.e. the attractive forces amongst the molecules. van der Waals was the first to deduce an equation of state in which all these factors are taken into account; according to van der Waals, we have

$$p = \frac{NK\theta}{v-b} - \frac{a}{v^2} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $b = 4 \times$ volume of the molecules, a defines the forces of cohesion.

In all subsequent modifications of this equation (Clausius, Dieterici, or D. Berthelot) the changes which have been proposed all relate to the influence of the cohesive forces; the part of the argument dealing with the finiteness of molecular volumes is generally left untouched.

But it has been found that the results of experiments do not agree with the predictions of theory if we regard a and b as absolute constants. Accordingly it has been proposed to regard both a and b as functions of volume and temperature.*

But before proceeding to these considerations, it is necessary to scrutinize whether the influence of finite molecular volumes is properly represented by the term b. From theoretical considerations, the conclusion has been reached that this is not the case. The argument is as follows :

According to Boltzmann's theory,

$$\text{the entropy } S = K \log W + C,$$

where K = Boltzmann's gas-constant, W = probability of the state. Let us now calculate the probability that a number N of molecules originally confined within the volume V_0 and possessing finite volumes, shall be contained in a volume. V. Neglecting the influence of internal forces, the probability for the first molecule is $\frac{V}{V_0}$, for the second molecule the probability is $\frac{V-\beta}{V_0-\beta}$, where $\beta = 8 \times$ volume of a single molecule, for when

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[†] Communicated by the Authors.

*Compare van der Waals, *Proc. Amst.* 1916 ; Van Laar, *Proc. Amst.* Vol. xvi, p. 44.

the first molecule is in position, the space enclosed by a concentric sphere of double the radius of the molecule will not be available for the second molecule. The available space is therefore $V-\beta$, whence the probability is $\frac{V-\beta}{V_0-\beta}$. Introducing similar considerations for the rest of the molecules, we have

$$W=\frac{V}{V_0}\cdot\frac{V-\beta}{V_0-\beta}\cdot\frac{V-2\beta}{V_0-2\beta}\cdots\frac{V-\overline{N-1}\beta}{V_0-\overline{N-1}\beta} \qquad \ldots \quad (2)$$

We are, of course, neglecting those cases in which partial overlapping of the regions occupied by two or more molecules occurs; for the number of such cases can at best be a small fraction of the total number. Even cases of actual association do not include these, for in that case, two discrete molecules become merged into one, without their outer surfaces being actually in contact.

From the relations $S = K \log W+C$

and
$$\left(\frac{\partial S}{\partial V}\right)_u=\frac{p}{\theta}$$
we can easily verify that

$$p=-\frac{K\theta}{\beta}\log\frac{V-n\beta}{V}$$

$$=-\frac{R\theta}{2b}\log\frac{V-2b}{V}\ (R=NK) \qquad \ldots \quad (3)$$

As a first approximation, when b is small compared to v, we obtain $p=\frac{NK\theta}{v}$ (Boyle-Charles-Avogadro Law), and as a second approximation we obtain

$$p=\frac{NK\theta}{v-b}\ \text{(van der Waals correction).}$$

We also note that

$$pV=NK\theta.\frac{x}{1-e^{-x}}\ \text{where}\ x=\frac{\beta p}{K\theta}. \qquad \ldots \quad (4)$$

To account for the influence of internal forces, we multiply, following the lead of Dieterici, the above expression (3) by $e^{-\frac{a}{NK\theta v}}$ having the same significance as before.

From this equation of state, we can easily verify the following results for the critical point :

Critical volume, $$V_c=\frac{2c}{e-1}\,b=3.166b,$$

$$K=\frac{NK\theta}{p_cV_c}=3.513.$$

The corresponding values of V_c from the van der Waals and the Dieterici equations are (3b, 2b) respectively, and of K are $\left(\frac{8}{3} = 2.66,\ \frac{e^2}{2}\ 3.695\right)$ respectively.

As a matter of fact, for the simpler gases, the value of 'K' obtained in this paper agrees better with the experimental results that the Dieterici value $\frac{e^2}{2}$ we have for oxygen* $K = 3.346$, for nitrogen† $K = 3.53$, for argon‡ $K = 3.424$, for xenon** $K = 3.605$. We need not consider the van-der-Waals value $\frac{8}{3}$, for it fails entirely.

The most serious drawback to Dieterici's equation is, according to Prof. Lewis (vide Lewis's Physical Chemistry, vol. ii. p. 117) that it makes b or the limiting volume $\frac{V_c}{2}$, while the limiting volume, obtained by the extrapolation of Cailletet-Mathias mean density line to the temperature $\theta = 0°\text{K}$ is about $\frac{V_c}{4}$. The value of b obtained in this paper, viz., $\frac{V_c}{3{\cdot}16}$ therefore agrees better with this value.

It is yet premature to predict what influence this investigation will have on the speculations concerning the variability of the volume of molecules with temperature. A more detailed investigation dwelling upon this point, and the application of the formula (4) to Amagat's (pv, p) curves, will be communicated shortly. Meanwhile we point out that the factor $e^{-\frac{a}{NK\theta v}}$ has been introduced into the expression for 'p' only as a provisional measure, though it is considered that this step, though not quite exact, is one in the right direction. In the next paper an attempt will be made to introduce energy into probability calculations.

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Note added in proof—On consulting the literature on the subject, we noticed that in several papers in the Amsterdam Proceedings (*vide* vol xv, p. 240et *seq.*), Dr. Keesom of Leyden had also made attempts to deduce the equation of state from Boltzmann's entropy principle. But, in the expression (2) for W, he introduces, before differentiation, an approximation in which terms up to second order in $\frac{b}{v}$ are retained only. In this way, he arrives at the van der Waals' form $v-b$ for the influence of finite molecular volumes. In obtaining our present equation of state (4), no such approximation has been made.

(M. N. Saha *and* S. N. Basu.)

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*Mathias and K. Onnes, *Proc. Amst.* Feb. 1911.

†Berthelot, *Bull. de la Soc. France de Phys.* 167 (1901)

‡Mathias, Onnes, and Crommelin, *Proc. Amst.* 1913, p. 960, Vol. xv.

**Paterson, Cripps, Whytlaw-Gray, Proc. Roy. Soc. Lond. A. lxxvi, p. 579 (1912).

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The Stress-Equations of Equilibrium

BY
SATYENDRANATH BASU.

(*Read April 6th,* 1919)

It was shown by Mitchell that the six stress-coefficients in an isotropic medium satisfy six equations of the type :

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial x^2} = 0, \quad \nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial y \partial z} = 0,$$

... ...

... ...

These equations however have not been used for solving the general problems of Elasticity. It is shown here, that the equations can be successfully integrated in the case of a semi-infinite body bounded by a plane. In the case of the sphere the equtions can be conveniently transformed, in a different form, which then admit of integration in an infinite series of spherical harmonics.

(1) *The semi-infinite solid bounded by* $z = 0$

The surface tractions X_z Y_z Z_z, are supposed to have given values over the plane $z = 0$

Consider the equations

$$\nabla^2 X_z + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial x \partial z} = 0, \quad \nabla^2 Z_z + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial z^2} = 0,$$

$$\nabla^2 Y_z + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial y \partial z} = 0.$$

Since $\odot$ is a harmonic function the general solution can be written as

$$X_z = -\frac{1}{2(1+\sigma)} z \frac{\partial \odot}{\partial x} + X_{z0},$$

$$Y_z = -\frac{1}{2(1+\sigma)} z \frac{\partial \odot}{\partial y} + Y_{z0}, \qquad \ldots \quad (1)$$

$$Z_z = -\frac{1}{2(1+\sigma)} z \frac{\partial \odot}{\partial z} + Z_{z0},$$

where X_{z0}., Y_{z0}, Z_{z0}, are harmonic functions which have given values $X_z Y_z Z_z$, over the plane $z = 0$.

The functions are therefore uniquely determined; they are in fact:

$$X_{z0} = \frac{1}{2\pi}\frac{\partial}{\partial z} \int\int \frac{X_z}{r}\, dxdy, \quad Y_{z0} = \frac{1}{2\pi}\frac{\partial}{\partial z} \int\int \frac{Y_z}{r}\, dxdy,$$

$$Z_{z0} = \frac{1}{2\pi}\frac{\partial}{\partial z} \int\int \frac{Z_z}{r}\, dxdy,$$

also since

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{\partial Z_z}{\partial z} = 0$$

we have from (1)

$$-\frac{1}{2(1+\sigma)} \cdot \frac{\partial \odot}{\partial z} + \frac{1}{2\pi}\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} \right) = 0,$$

where

$$L = \int\int \frac{X_z}{r}\, dxdy, \quad M = \int\int \frac{Y_z}{r}\, dxdy, \quad N = \int\int \frac{Z_z}{r}\, dxdy,$$

and

$$\odot = \frac{1+\sigma}{\pi}\left[\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} \right].$$

Thus X_z, Y_z, Z_z, $\odot$ are all determined.

The solution may be afterwards completed, and U, V, W found out as in Cerrutti's method.

(2) *The problem of the sphere.*

Consider the three equations

$$\nabla^2 X_x + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial x^2} = 0,$$

$$\nabla^2 X_y + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial x \partial y} = 0,$$

$$\nabla^2 X_z + \frac{1}{1+\sigma}\frac{\partial^2 \odot}{\partial x \partial z} = 0.$$

Multiplying by x, y, z, and adding we have

$$\left[x\nabla^2 X_x + y\nabla^2 X_y + z\nabla^2 X_z \right] + \frac{1}{1+\sigma}\left[x\frac{\partial^2\Theta}{\partial x^2} + y\frac{\partial^2\odot}{\partial x\partial y} + z\frac{\partial^2\odot}{\partial x\partial z} \right] = 0,$$

or,

$$\nabla^2(xX_x + yX_y + zX_z) + \frac{1}{1+\sigma}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right)\frac{\partial\odot}{\partial x} - 2\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} \right) = 0.$$

Now since

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0,$$

we have

$$\nabla^2(rX_r) + \frac{1}{1+\sigma}\frac{\partial}{\partial x}\left(r\frac{\partial\odot}{\partial r} - \odot \right) = 0.$$

Similarly we have

$$\left.\begin{aligned} &\nabla^2(rY_r) + \frac{1}{1+\sigma}\frac{\partial}{\partial y}\left(r\frac{\partial\odot}{\partial r} - \odot \right) = 0, \\ &\nabla^2(rZ_r) + \frac{1}{1+\sigma}\frac{\partial}{\partial z}\left(r\frac{\partial\odot}{\partial r} - \odot \right) = 0, \end{aligned}\right\} \quad (2)$$

where $r\dfrac{\partial\odot}{\partial r} - \odot$ is also a harmonic function.

The form of the equations is exactly similar to the preceding equations.

(3) *The sphere of radius* a, *has given tractions* X_r, Y_r, Z_r, *over the surface.*

Assuming $$\Theta = \sum_n \Theta_n,$$

where $\odot_n$ is a solid homogenoous harmonic of the nth degree,

$$r\frac{\partial\odot}{\partial r} - \odot = \Sigma\,(x-1)\,\odot_x$$

also remembering that

$\nabla^2(r^2-a^2)F_n = 2(2n+3)F_n$, where F_n is a solid homogeneous harmonic of the nth degree, we see the solutions of the equation (2) can be expressed in the form

$$rX_r = -\frac{1}{2(1+\sigma)}\,\Sigma\,\frac{(n-1)}{2n+1}\,(r^2-a^2)\,\frac{\partial\odot_n}{\partial x} + aX_{r0},$$

$$rY_r = -\frac{1}{2(1+\sigma)}\,\Sigma\,\frac{n-1}{2n+1}(r^2-a^2)\frac{\partial\odot_n}{\partial y}+aY_{r0},$$

$$rZ_r = -\frac{1}{2(1+\sigma)}\,\Sigma\,\frac{n-1}{2n+1}(r^2-a^2)\frac{\partial\odot_n}{\partial z}+aZ_{r0},$$

where X_{r0}, Y_{r0}, Z_{r0}, are harmonic functions which have given values over the surface of the sphere $r = a$ and hence are completely determined. If

$$X_{r0} = \Sigma . X_n,\ Y_{r0} = \Sigma\, Y_n,\ Z_{r0} = \Sigma\, Z_n;$$

we have

$$rX_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\,\Sigma\,\frac{n-1}{2n+1}\,\frac{\partial\odot_m}{\partial x}+a\,\Sigma\,X_n$$

$$rY_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\,\Sigma\,\frac{n-1}{2n+1}\,\frac{\partial\odot_n}{\partial y}+a\,\Sigma\,Y_n$$

$$rZ_r = -\frac{(r^2-a^2)}{2(1+\sigma)}\,\Sigma\,\frac{n-1}{2n+1}\,\frac{\partial\odot_n}{\partial z}+a\Sigma\,Z_n.$$

Again

$$\frac{\partial}{\partial x}(rX_r)+\frac{\partial}{\partial y}(rY_r)+\frac{\partial}{\partial z}(rZ_r)$$

$$=x\left[\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right]+y\left[\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right]$$

$$+Z\left[\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right]+X_x+Y_y+Z_z=\odot,$$

it follows that

$$\odot = -\frac{1}{(1+\sigma)}\,\Sigma\,\frac{n-1}{(2n+1)}\left[x\frac{\partial\odot_n}{\partial x}+y\frac{\partial\odot_n}{\partial y}+z\frac{\partial\odot_n}{\partial z}\right]$$

$$+a\,\Sigma\left(\frac{\partial x_n}{\partial x}+\frac{\partial y_n}{\partial y}+\frac{\partial z_n}{\partial z}\right)$$

or

$$\odot = -\frac{1}{1+\sigma}\,\Sigma\,\frac{n(n+1)}{2n+1}\,\odot_n+a\Sigma\left(\frac{\partial X_n}{\partial x}+\frac{\partial Y_n}{\partial y}+\frac{\partial Z_n}{\partial z}\right).$$

So that

$$\Sigma\odot_n \left[\frac{(1+\sigma)(2n+1)+n(n-1)}{(2x+1)(1+\sigma)} = a\Sigma\psi_n, \right.$$

where

$$\psi_n = \frac{\partial X_{n+1}}{\partial x} + \frac{\partial Y_{n+1}}{\partial y} + \frac{\partial Z_{n+1}}{\partial z}.$$

So that

$$\odot_n = \frac{a(2n+1)(1+\sigma)}{(1+\sigma)(2n+1)(+n(n-1)} \cdot \psi_n.$$

Thus X_r, Y_r, Z_r, $\odot$ are all determined in terms of the known value of X_r, Y_r, Z_r, on the surface.

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On the Herpolhode

BY

SATYENDRANATH BASU

(*Read August 31st, 1919*)

M. de Sparre has shown that the horpolhode of Poinsot contains no point of inflexion; his proof was based upon the properties of elliptic functions. Various other proofs of the theorem have since been given by Mannheim, Saint-Germain[2], Routh[3], Le-Cornu[4] and others.

The following simple proof of the theorem is based solely on the dynamical considerations of the Poinsot motion; hence it might appear interesting and instructive.

Let A, B, C be the three principal moments of inertia of a body fixed at a point 0; let 0I and 0I′ be the two instantaneous axes, at time t and $t+dt$, I, I', etc., on the invariable plane trace out the horpolhode, length of 0I, 0I′, etc. being proportional to the resulting angular velocity at all times.

At a point of inflexion of the horpolhode, there will be a stationary tangent, i.e., the total change of the vector II' will be then along its own length.

Remarking that II′ has components proportional to $\dot{w}_1\, dt$, $\dot{w}_2\, dt$, $\dot{w}_3\, dt$, along the three moving principal axes, the condition reduces to

$$\frac{\ddot{w}_1-\dot{w}_2w_3+\dot{w}_3w_2}{\dot{w}_1}=\frac{\ddot{w}_2-\dot{w}_3w_1+\dot{w}_1w_3}{\dot{w}_2}=\frac{\ddot{w}_3-\dot{w}_1w_2+\dot{w}_2w_1}{\dot{w}_3}=\lambda \text{ say.} \quad \ldots \quad (1)$$

We have from Euler's equations

$$A\dot{w}_1=(B-C)w_2w_3,$$

$$B\dot{w}_2=(C-A)w_3w_1$$

$$C\dot{w}_3=(A-B)w_1w_2;$$

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[1] An exactly similar proof of the theorem discussed in this paper has been given by Prof. W. Van der Woude in a paper entitled "Over de Herpoloide Van Poinsot", published in Nieuw Archief voor Wiskunde, Tweede Reeks, Deel XII, Eerste Stuk, pp. 94, 95 (1919). It appears however from an enquiry that this journal was not received in the library of the Society before the beginning of November, 1919 —S. K. B.

[2] Compte Rendu—1885

[3] Advanced Dynamics.

[4] Bulletin de Soc. Math. de France—1906.

from which

$$A\ddot{w}_1 = (B-C)(\dot{w}_2 w_3+\dot{w}_3 w_2)$$

$$B\ddot{w}_2 = (C-A)(\dot{w}_3 w_1+\dot{w}_1 w_3)$$

$$C\ddot{w}_3 = (A-B)(\dot{w}_1 w_2+\dot{w}_2 w_1)$$

Substituting in (1) we have

$$\frac{\dot{w}_3}{w_3}(A+B+C)-\frac{\dot{w}_2}{w_2}(C+A-B) = \lambda(B-C)$$

and two similar equations.

So that

$$\frac{\dot{w}_1}{w_1}(B-C)(B+C-A)+\frac{\dot{w}_2}{w_2}(C-A)(C+A-B)+\frac{\dot{w}_3}{w_3}(A-B)(A+B-C) = 0$$

But remembering that

$$Aw_1\dot{w}_1+Bw_2\dot{w}_2+Cw_3\dot{w}_3 = 0$$

$$A^2w_1\dot{w}_1+B^2w_2\dot{w}_2+C^2w_3\dot{w}_3 = 0$$

we have

$$\frac{w_1\dot{w}_1}{BC(B-C)} = \frac{w_2\dot{w}_2}{CA(C-A)} = \frac{w_3\dot{w}_3}{AB(A-B)}$$

hence substituting, we get as a necessary condition for the point of inflexion

$$\frac{BC(B-C)^2(B+C-A)}{w_1^2}+\frac{CA(C-A)^2(C+A-B)}{w_2^2}+\frac{AB(A-B)^2(A+B-C)}{w_3^2} = 0$$

which is obviously impossible because $B+C-A$, $C+A-B$, $A+B-C$, are all positive, so that the left-hand side is an essentially positive quantity.

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ON THE EQUATION OF STATE.

To the Editors of the Philosophical Magazine.

Sirs, —

In a paper published in the 'Philosophical Magazine' for August 1918 it has been shown that from Boltzmann's theory of entropy we can arrive at an equation of state

$$p = -\frac{R\theta_c}{2b} \log\left(1 - \frac{2b}{V}\right) = e^{-\frac{a}{NK\theta v}} \qquad \dots \quad (1)$$

Amongst other applications of this theorem it was shown that the value of the critical constant $K = \frac{R\theta_c}{p_c V_c}$ would be $\frac{2e}{e-1} = 3.53$, instead of van der Waals' $\frac{8}{3}$ and Dieterici's $\frac{e^2}{2} = 3.69$.

A table was given showing that in the case of the most of the elementary gases the value 3.53 corresponded better with experimental results than either $\frac{8}{3}$ or 3.69.

The list comprised He, N_2, O_2, Xe, but not Hydrogen, about which the experimental data were not satisfactory.

Recently the critical data for Hydrogen have been re-determined in the Laboratory of Prof. K. Onnes (vide Proc. K. Akad. Wetenschappen. Amsterdam, Vol. xx. 1917).

It has been found that

$$\theta_c = 33^\circ.18\text{K}, \quad d_c = .0310, \quad p_c = 12.8 \text{ atmospheres;}$$

from these data, $K = 3.35.$

It is superfluous to add that this value of K is in much better agreement with the value 3.53 than with either the value $\frac{8}{3} = 2.66$ or $\frac{e^2}{2} = 3.69$, thus corroborating the belief expressed in the aforesaid paper that the equation (1), though not final, marks a step in the right direction.

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Megh Nad Saha.
Satyendra Nath Basu.

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*On the Deduction of Rydberg's Law from the Quantum Theory of Spectral Emission.** *By* SATYENDRA NATH BASU, *M. Sc., University Lecturer in Physics, University College of Science, Calcutta.*

It is well known that Rutherford's model of the atom has been fruitful in explaining many facts connected with atomic radiation. In the simplest case of hydrogen, with a nucleus consisting of a single positive charge, and an electron, Dr. Bohr has successfully applied the quantum theory to explain the Balmer series of hydrogen spectra. The mathematical problem of finding the spectral series for any atomic system has since been clearly formulated by Sommerfeld, and the quanta condition has been generalized in a form suitable for system with any number of degrees of freedom. If $q_1, q_2, q_3, \ldots q_n$ are co-ordinates to fix the position of the electron responsible for emission, and $p_1, p_2, p_3, \ldots, p_n$ are the corresponding generalized momenta, any statical path, according to Sommerfeld, is characterized by the conditions

$$\int p_1 dq_1 = n_1 h, \quad \int p_2 dq_2 = n_2 h, \quad \int p_n dq_n = n_n h$$

where n's are whole numbers and h is Planck's constant, the integral being extended generally over the complete orbit. The radiation is supposed to take place when the electron jumps from one statical path to another. The difference in energy, at the same time, flows away in the form of a homogeneous radiation of frequency ν, which can be calculated from the Bohr equation $h\nu = W_1 - W_2$. Sommerfeld has successfully applied this conception in explaining the fine structure of hydrogen lines. It is clear, however, that the problem of theoretically calculating the spectrum of any atom other than hydrogen is beset with difficulties of a formidable nature. It is exactly analogous to the dynamical problem of "n" bodies, where only in favourable cases we are able to find solutions. Nevertheless, from a purely experimental standpoint, we know that the visible radiation from any element can be classified in definite series. The frequency of any line in the series can be expressed as a difference of two terms, each of which has the form

$$\frac{N}{(m+\alpha+\beta/m^2)^2},$$

where m is a whole number and α and β are two constants depending upon the element and the nature of the series. So that if we are to explain the formation of the series

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* Communicated by the Author.

+Bohr, Phil. Mag. July 1913.

**Sommerfeld, Ann. der Physik, li (1916).

from theoretical considerations following Bohr and Sommerfeld we must look upon each member multiplied by "h" as giving the energy of the atomic system when the radiating electron moves in a definite statical path. The complexity of the inner atomic field under which the radiating electron moves is to be looked upon as bringing in the terms involving α and β. So it seems interesting to see what will be the corresponding expression for energy in a system by which the complex nature of the internal field may be approximately represented. In the case of any atom we have, in general, a condensed nuclear charge of $+ne$ (where n is the atomic number) surrounded by rings of electron at different distances. The number of electrons in total must be also equal to n in order to secure that the atom is electrically neutral in the ordinary state.

In X-ray emission the electron displaced comes from the inner rings; in the case of visible radiation, however, we have reasons to think that the displaced electron responsible for radiation comes from the outermost ring—the valency electrons, as they have been designated by Sommerfeld. When excited for radiation, we can suppose that the electron in the outermost ring is removed to a greater distance from the centre than the others, so that the force acting upon it may be regarded as the resultant of the various forces exerted by the central charge and the remaining electrons. The potential at any point can be regarded as given by

$$-\frac{ne^2}{r}+e^2 \sum_{s=n-1} \frac{1}{r_s}$$

where r is the distance from the centre and r_s is the distance from the s-th electron. If we neglect the influence of the moving electron upon the arrangement of the others surrounding the nucleus, it is clear that the potential can be approximately represented as $-\frac{e^2}{r}+\frac{L\cos\theta}{r^2}$. The resultant field might be looked upon as due to a single positive charge together with a doublet of strength L in a certain fixed direction which we take as our Z-axis. If we neglect the disturbing effect of the outer electron, L may be taken to be approximately fixed in direction and in magnitude in the small interval of time during which the active emission takes place.

We may, therefore, take as our model a system consisting of a positive charge and a doublet of strength L. We proceed to calculate the energy in a statical path on the above simplified hypothesis.

The kinetic energy of the moving electron is obviously

$$T=\frac{1}{2}\left[m\dot{r}^2+mr^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\phi}^2\right];$$

the potential energy

$$V = -\frac{e^2}{r} + \frac{eL\cos\theta}{r^2}.$$

Two integrals can be at once written down :

$$mr^2 \sin^2\theta\dot{\phi} = c_1,$$

$$m[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2] - \frac{2e^2}{r} + \frac{2Le\cos\theta}{r^2} = -W.$$

To get another integral, we write

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\dot{\phi}^2 = \frac{eL\sin\theta}{r^2},$$

or

$$mr^2\dot{\theta}\frac{d}{d\theta}(mr^2\dot{\theta}) - \frac{c_1^2\cos\theta}{\sin^3\theta} - meL\sin\theta = 0.$$

Integrated, it gives

$$(mr^2\dot{\theta})^2 + \frac{c_1^2}{\sin^2\theta} + 2meL\cos\theta = c_2.$$

The expressions for three impulses mr, $mr^2\dot{\theta}$ and $mr^2\sin^2\theta\phi$ can now be written down : in terms of the constants of integration we have

$$mr^2\sin^2\theta\dot{\phi} = c_1$$

$$mr^2\dot{\theta} = \sqrt{c_2 - \frac{c_1^2}{\sin^2\theta} - 2meL\cos\theta}$$

$$mr = \frac{1}{\sigma}\sqrt{-Wmr^2 + 2me^2r - c_2}.$$

The quanta conditions can be written down as

$$\int mr^2\sin^2\theta\dot{\phi}\,d\phi = n_1h, \qquad \dots \quad (1)$$

$$\int mr^2\dot{\theta}\,d\theta = n_2h, \qquad \dots \quad (2)$$

$$\int m\dot{r}dr = n_3h, \qquad \dots \quad (3)$$

$-W$ being twice the total energy of the system. The integrals are to be extended over the whole range within which the expression within the square root remains positive.

The integration.

From (1) we have obviously

$$c_1 = \frac{n_1 h}{2\pi};$$

of the two remaining expressions, (3) can be integrated most easily : in fact,

$$n_3 h = \int \frac{1}{r} \sqrt{-Wmr^2 + 2me^2 r - c_2} \; dr$$

gives after integration

$$n_3 h = 2\pi \left[e^2 \left(\frac{m}{W} \right)^{\frac{1}{2}} - c_2^{\frac{1}{2}} \right].$$

The second integral can be written as

$$\frac{n_2 h}{2} = I = \int \sqrt{(c_2 - c_1^2) - c_2 x^2 - 2meLx(1-x^2)} \; \frac{dx}{1-x^2}$$

by putting $\cos\theta = x$.

The right-hand side is to be integrated throughout the region, when the cubic remains positive. It cannot be integrated in finite terms; an approximation suitable for our purpose can, however, be made, assuming $2meL$ to be small compared with $(c_2 - c_1^2) = A$. To see what this means we are to remember that $c_2 - c_1^2$ is of the dimension of h^2; so that $2meL$ must be small compared with h^2, or L must be small compared with $\frac{h^2}{me}$. Now, if α, $-\beta$, and γ are taken as the three roots of the cubic, the cubic can be written down as

$$D(\gamma - x)(\alpha - x)(x + \beta),$$

where γ is the greatest of the positive roots and $D = 2meL$. The limits of the integral are obviously α and $-\beta$,

$$I = \int_{-\beta}^{\alpha} \sqrt{A - c_2 x^2 - Dx(1-x^2)} \; \frac{dx}{1-x^2}.$$

Hence

$$\frac{\partial I}{\partial D} = \frac{1}{2} \int_{-\beta}^{\alpha} \frac{-x\,dx}{\sqrt{D}[(\gamma-x)(\alpha-x)(x+\beta)]^{\frac{1}{2}}},$$

or

$$\frac{\partial I}{\partial D} = \frac{1}{2\sqrt{D}} \left[\int_{-\beta}^{\alpha} \frac{(\gamma-x)^{\frac{1}{2}}dx}{(\alpha-x)^{\frac{1}{2}}(\beta+x)^{\frac{1}{2}}} - \gamma \int_{-\beta}^{\alpha} \frac{dx}{(\gamma-x)^{\frac{1}{2}}(\alpha-x)^{\frac{1}{2}}(x+\beta)^{\frac{1}{2}}} \right].$$

Supposing

$$\alpha - x = (\alpha+\beta)\cos^2\theta,$$
$$\beta + x = (\alpha+\beta)\sin^2\theta,$$

we get

$$\frac{\partial I}{\partial D} = \frac{1}{2\sqrt{D}}\left[2\int_0^{\pi/2}[\gamma+\beta-(\alpha+\beta)\sin^2\theta]^{\frac{1}{2}}\,d\theta - 2\gamma\int_0^{\pi/2}\frac{d\theta}{[(\gamma+\beta)-(\alpha+\beta)\sin^2\theta]^{\frac{1}{2}}}\right]$$

$$= \frac{1}{2\sqrt{D}}\left[(\gamma+\beta)^{\frac{1}{2}}E\left(\pi/2, \sqrt{\frac{\alpha+\beta}{\gamma+\beta}}\right) - \frac{\gamma}{(\gamma+\beta)^{\frac{1}{2}}}K\left(\pi/2, \sqrt{\frac{\alpha+\beta}{\gamma+\beta}}\right)\right],$$

where E and K are the usual elliptic integrals, defined by

$$E(\pi/2, k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$$

$$K(\pi/2, k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

On the assumption that D is small, we have

$$\gamma = \frac{c_2}{D} + \frac{D(c_2-A)}{c_2^2},$$

$$\alpha = \sqrt{\frac{A}{c_2}} - \frac{D}{2c_2^2}(c_2-A),$$

$$\beta = \sqrt{\frac{A}{c_2}} + \frac{D}{2c_2^2}(c_2-A).$$

So that, expanding E and K and making necessary approximations, we have finally

$$\frac{\partial I}{\partial D} = \frac{\pi D}{4c_2^{5/2}}\left[c_2 - \frac{3A}{2}\right].$$

Also

$$I = \int\sqrt{(c_2-c_1^2)c_2x^2}\,\frac{dx}{1-x^2} = \pi\,[c_2^{\frac{1}{2}} - c_1].$$

So that we have

$$\frac{n_2h}{2} = \pi(c_2^{\frac{1}{2}} - c_1) + \frac{\pi}{16}\,\frac{(2meL)^2}{c_2^{5/2}}\,(3c_1^2 - c_2).$$

Now, collecting all the quanta conditions, we have

$$c_1 = \frac{n_1h}{2\pi},$$

$$c_2^{\frac{1}{2}} - c_1 + \frac{(meL)^2}{4c_2^{5/2}}\,(3c_1^2 - c_2) = \frac{n_2h}{2\pi},$$

$$e^2\left(\frac{m}{W}\right)^{\frac{1}{2}} - c_2^{\frac{1}{2}} = \frac{n_3h}{2\pi}.$$

So that we have

$$c_2^{\frac{1}{2}} + \frac{(meL)^2}{4c_2^{5/2}}\left(\frac{3n_1h_1^2}{4\pi^2} - c_2\right) = \frac{(n_1+n_2)h}{2\pi}$$

and

$$e^2\left(\frac{m}{W}\right)^{\frac{1}{2}} = \frac{n_3h}{2\pi} + c_2^{\frac{1}{2}}.$$

Assuming

$$c_2^{\frac{1}{2}} = \frac{yh}{2\pi}$$

we have

$$\frac{yh}{2\pi} + \frac{(meL)^2(2\pi)^5}{4y^5h^5}\left[\frac{3n_1^2\,h^2}{4\pi^2} - \frac{y^2h^2}{4\pi^2}\right] = \frac{(n_1+n_2)}{2\pi}\,h$$

or

$$y + \left(\frac{2\pi^2\,meL}{h^2}\right)^2 \frac{1}{y^5}\,(3n_1^2 - y^2) = n_1 + n_2.$$

Calling

$$\left(\frac{2\pi^2meL}{h^2}\right)^2 = \alpha,$$

we have

$$e^2\left(\frac{m}{W}\right)^{\frac{1}{2}} = \frac{h}{2\pi}\,(n_3 + y),$$

where

$$y + \frac{\alpha}{y^5}\,(3n_1^2 - y^2) = n_1 + n_2\,;$$

we have approximately

$$y = (n_1+n_2) + \frac{\alpha}{(n_1+n_2)^3}\left[1-\frac{3n}{(n_1+n_2)^2}\right].$$

So that

$$W = \frac{4\pi^2e^4m}{h^2\left[n_1+n_2+n_3+\frac{\alpha}{(n_1+n_2)^3}\left(1-\frac{3n_1^2}{(n_1+n_2)^2}\right)\right]^2}.$$

So that the energy in the statical path

$$= -\frac{2\pi^2e^4m}{h^2\left[n_1+n_2+n_3+\frac{\alpha}{(n_1+n_2)^3}\left(1-\frac{3n}{(n_1+n_2)^2}\right)\right]^2}$$

if we suppose that the spectrum is due to ionized atoms in which the field can be approximately represented as a central charge of $E = pe$ and a doublet of different strength L'.

We have, by a similar reasoning, energy in a statical path

$$= -\frac{2\pi^2}{h^2}\frac{e^2E^2m}{\left\{n_1+n_2+n_3+\frac{\alpha'}{(n_1+n_2)^3}\left(1-\frac{3n_1^2}{(n_1+n_2)^2}\right)\right\}^2}$$

$$= \frac{-p^2Nh}{\left\{n_1+n_2+n_3+\frac{\alpha'}{(n_1+n_2)^3}\left(1-\frac{3n_1^2}{(n_1+n_2)^2}\right)\right\}^2}$$

where N = Rydberg number.

As a result of numerous investigations on the nature of the spectral series, it has been shown that for many elements the different series can be grouped according to the following schemes :

P-Series :

$$\left.\begin{array}{l}\nu = (1.5,\ s)-(m,\ p_1)\\ \qquad\qquad\quad -(m,\ p_2)\end{array}\right\} \qquad m = 2, 3, 4\ldots$$

II. Subordinate Series :

$$\nu = (2,\ p_1)-(m+\cdot 5,\ s)$$

$$m = 2, 3, 4\ldots$$

$$\nu = (2,\ p_2)-(m+\cdot 5,\ s)\cdot$$

1st Subordinate Series :

$$\nu = (2,\ p_1)-(m,\ d).$$

Companion ;

$$\nu = (2,\ p_1)-(m,\ d')$$

$$m = 3, 4, 5.$$

$$\nu = (2,\ p_2)-(m,\ d'').$$

Symbol (m, f) stands for $\frac{N}{(m+f)^2}$ according to Rydberg, and $\frac{N}{\left(m+f+\frac{\phi}{m^2}\right)^2}$ according to Ritz.

The frequency of the lines emitted thus appears as the difference of two terms, each of which is to be regarded as corresponding to the energy in a definite statical path, on the Quantum theory. Sommerfeld* has recently given reasons for assuming that in any series of statical orbits corresponding to the different kinds of s, p, and d terms the azimuthal quanta generally preserve a certain definite value, whereas the radial quanta can have all values from 0 to ∞; he thus shows that in s, p and d terms the azimuthal quanta generally have values 1, 2, and 3 respectively.

Making the above assumptions in our formula, we see that the expression of the energy comes out as $-\frac{Nh}{(n_1+n_2+n_3+A)^2}$ in the same form as required by the Rydberg formula. The constants A, however, depends only upon n_1 and n_2; it diminishes for increasing values of the azimuthal quanta; so that they decrease progressively in the s, p, and d terms. Moreover, our form shows that A depends upon n_1 and n_2 separately, so that for the same value of n_1+n_2 we may have different values of the constant. Thus, if we suppose $n_1+n_2 = 1$ we have two values corresponding to the vaues 1, 0 and 0, 1; for $n_1+n_2 = 2$ we have three values; and so on. Thus we see, even on Sommerfeld's assumption, for the constancy of the azimuthal quanta we shall have two different s, three different p, four different d terms. At least two different values of p and three different values of d seem to be required by the series formula, which is essential for the explanation of doublets and triplets of constant frequency difference**. We thus see that our model serves at least as a qualitative explanation of the following facts :

(1) The progressive decrease of the characteristic numbers in the s, p, and d terms,

(2) The existence of different sets of s, p, and d terms for the same element.

It is clear, however, that our simplified assumption will not fit in any actual case exactly. The complex nature of the internal field can in no case be properly represented by a simple term, $\frac{L \cos \theta}{r^2}$, in the Potential. Moreover, we have reason to believe that the internal arrangement of the electrons itself will be influenced, in a large measure, by the motion of the outer electron, which we have neglected in our formula. In fact, Landé† has tried in a recent paper to take account of this disturbance in the

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* Sommerfeld, Verh. d. Phys. Ges. May 1919.

** If we exclude the case $n_2 = 0$—*i.e.*, if we assume that the motion in a plane containing the axis of the doublet is excluded—we get the proper number of s, p, d terms as observed in the case of the alkali metals and the doublet system of alkaline earths.

† Landé, Phys. Zeit. 1909.

comparative simple case of the helium series. But at the same time, it is hoped that the calculation, in this comparatively simple case, will serve to illustrate at least some general principles at which we have arrived by an experimental study of the spectral series.

SUMMARY

In this paper an attempt has been made to deduce the laws of regularity in the spectral series of elements on the basis of Bohr's quantum theory of spectral emission. Starting from Sommerfeld's assumption that the ordinary line-spectra of elements are due to the vibration of one outer electron (the valency electron), it has been shown that the field of the nucleus and the remaining $(n-1)$ electrons may be represented by the Potential $V=-\frac{e^2}{r}+\frac{eL\cos\theta}{r^2}$ i.e., the field due to a single charge plus a doublet of strength L. The axis of the doublet is variable, but the emission is supposed to take place so quickly that in that short time the axis does not appreciably change.

The quanta conditions have been applied according to Sommerfeld's rule, $nh=\int p_i\,dq_i$, and the energy of the system has been reduced to the quanta numbers. The energy comes out in the form

$$W=-\frac{Nh}{(n+n_3+z)^2},\quad n+z=y$$

in the paper, where n_3 is the radial quantum, n is the azimuthal quantum, and z is given by an equation of the sixth degree, involving only the azimuthal quantum, and is a function of n only.

It has been next shown that if, in accordance with Sommerfeld's principle, we assume $n=1$ for the s-orbits, $n=2$ for the p-orbits, $n=3$ for the d-orbits, $n=4$ for the b-orbits, then, with a very simple assumption, we obtain a single value for the energy of the s orbits, a double value for the energy in the p-orbit, a treble value for the d-orbit. Then, applying Bohr's law $h\nu=W_n-W_{n'}$, we arrive at Rydberg's laws of the regularity in spectral series, in the case of the alkali metals.

Exact calculations are not tried on account of the uncertainty of the value of L; but it has been pointed out that the values of s, (p_1, p_2), (d_1, d_2, d_3) progressively decrease, as is actually the case.

If the value of L be supposed to vary with n_3 the radial quantum, then probably the above calculations would lead to Ritz's law.

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Plancks Gesetz und Lichtquantenhypothese.

Von **Bose** (Dacca-University, Indien).

(Eingegangen am 2. Juli 1924.)

Der Phasenraum eines Lichtquants in bezug auf ein gegebenes Volumen wird in „Zellen“ von der Größe h^3 aufgeteilt. Die Zahl der möglichen Verteilungen der Lichtquanten einer makroskopisch definierten Strahlung unter diese Zellen liefert die Entropie und damit alle thermodynamischen Eigenschaften der Strahlung.

Plancks Formel für die Verteilung der Energie in der Strahlung des schwarzen Körpers bildet den Ausgangspunkt für die Quantentheorie, welche in den letzten 20 Jahren entwickelt worden ist und in allen Gebieten der Physik reiche Früchte getragen hat. Seit der Publikation im Jahre 1901 sind viele Arten der Ableitung dieses Gesetzes vorgeschlagen worden. Es ist anerkannt, daß die fundamentalen Voraussetzungen der Quantentheorie unvereinbar sind mit den Gesetzen der klassischen Elektrodynamik. Alle bisherigen Ableitungen machen Gebrauch von der Relation

$$\varrho_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} E,$$

d. h. von der Relation zwischen der Strahlungsdichte und der mittleren Energie eines Oszillators, und sie machen Annahmen über die Zahl der Freiheitsgrade des Äthers, wie sie in obige Gleichung eingeht (erster Faktor der rechten Seite). Dieser Faktor konnte jedoch nur aus der klassischen Theorie hergeleitet werden. Dies ist der unbefriedigende Punkt in allen Ableitungen, und es kann nicht wundernehmen, daß immer wieder Anstrengungen gemacht werden, eine Ableitung zu geben, die von diesem logischen Fehler frei ist.

Eine bemerkenswert elegante Ableitung ist von Einstein angegeben worden. Dieser hat den logischen Mangel aller bisherigen Ableitungen erkannt und versucht, die Formel unabhängig von der klassischen Theorie zu deduzieren. Von sehr einfachen Annahmen über den Energieaustausch zwischen Molekülen und Strahlungsfeld ausgehend, findet er die Relation

$$\varrho_\nu = \frac{\alpha_{mn}}{e^{\frac{\varepsilon_m - \varepsilon_n}{kT}} - 1}.$$

Indessen muß er, um diese Formel mit der Planckschen in Übereinstimmung zu bringen, von Wiens Verschiebungsgesetz und Bohrs Korrespondenzprinzip Gebrauch machen. Wiens Gesetz ist auf die klassische Theorie gegründet, und das Korrespondenzprinzip nimmt an, daß die Quantentheorie mit der klassischen Theorie in gewissen Grenzfällen übereinstimme.

In allen Fällen scheinen mir die Ableitungen nicht genügend logisch gerechtfertigt. Dagegen scheint mir die Lichtquantenhypothese in Verbindung mit der statistischen Mechanik (wie sie durch Planck den Bedürfnissen der Quantentheorie angepaßt worden ist) für die Ableitung des Gesetzes unabhängig von der klassischen Theorie hinreichend zu sein. Im folgenden will ich die Methode kurz skizzieren.

Die Strahlung sei in das Volumen V eingeschlossen und ihre Gesamtenergie E sei gegeben. Es seien verschiedene Sorten von Quanten von der jeweiligen Anzahl N_s und Energie $h\nu_s$ vorhanden ($s = 0$ bis $s = \infty$). Die Totalenergie E ist dann

$$E = \sum_s N_s h\nu_s = V\int \varrho_\nu d\nu. \tag{1}$$

Die Lösung des Problems verlangt dann die Bestimmung der N_s, welche ϱ_ν bestimmen. Wenn wir die Wahrscheinlichkeit für jede durch beliebige N_s charakterisierte Verteilung angeben können, dann wird die Lösung durch die Bedingung bestimmt, daß diese Wahrscheinlichkeit bei Wahrung der Nebenbedingung (1) ein Maximum sein soll. Diese Wahrscheinlichkeit wollen wir nun aufsuchen.

Das Quantum hat ein Moment vom Betrage $\frac{h\nu_s}{c}$ in der Richtung seiner Fortbewegung. Der Momentanzustand des Quantums wird charakterisiert durch seine Koordinaten x, y, z und die zugehörigen Momente $p_x\, p_y\, p_z$; diese sechs Größen können als Punktkoordinaten in einem sechsdimensionalen Raum aufgefaßt werden, wobei wir die Beziehung haben

$$p_x^2 + p_y^2 + p_z^2 = \frac{h^2\nu^2}{c^2},$$

vermöge welcher der genannte Punkt auf einer durch die Frequenz des Quants bestimmten Zylinderfläche zu bleiben gezwungen ist. Zum

Frequenzbereich $d\nu_s$ gehört in diesem Sinne der Phasenraum

$$\int dx\,dy\,dz\,dp_x\,dp_y\,dp_z = V.4\pi\left(\frac{h\nu}{c}\right)^2\frac{h\,d\nu}{c} = 4\pi\frac{h^3\nu^2}{c^3}\,V d\nu.$$

Wenn wir das gesamte Phasenvolumen in Zellen von der Größe h^3 einteilen, gehören zum Frequenzbereich $d\nu$ also $4\pi\,V\frac{\nu^2}{c^3}\,d\nu$ Zellen. In bezug auf die Art dieser Einteilung kann nichts Bestimmtes gesagt werden. Indessen muß die Gesamtzahl der Zellen als die Zahl der möglichen Anordnungen eines Quants in dem gegebenen Volumen angesehen werden. Um der Tatsache der Polarisation Rechnung zu tragen, erscheint es dagegen geboten, diese Zahl noch mit 2 zu multiplizieren, so daß wir für die Zahl der zu $d\nu$ gehörigen Zellen $8\pi\,V\frac{\nu^2\,d\nu}{c^3}$ erhalten.

Nun ist es einfach, die thermodynamische Wahrscheinlichkeit eines (makroskopisch definierten) Zustandes zu berechnen. Es sei N^s die Zahl der zum Frequenzbereich $d\nu^s$ gehörigen Quanten. Auf wie viele Arten können diese auf die zu $d\nu^s$ gehörigen Zellen verteilt werden? Sei p_0^s die Zahl der vakanten Zellen, p_1^s die Zahl derer, die ein Quant enthalten, p_2^s die Zahl der Zellen, die zwei Quanten enthalten usf. Die Zahl der möglichen Verteilungen ist dann

$$\frac{A^s!}{p_0^s!\,p_1^s!\ldots}, \quad \text{wobei} \quad A^s = \frac{8\pi\nu^2}{c^3}\,d\nu^s,$$

und wobei

$$N^s = 0\,.\,p_0^s + 1\,.\,p_1^s + 2\,p_2^s\ldots$$

die Zahl der zu $d\nu^s$ gehörigen Quanten ist.

Die Wahrscheinlichkeit des durch sämtliche p_r^s definierten Zustandes ist offenbar

$$\prod_s \frac{A^s!}{p_0^s!\,p_1^s!\ldots}.$$

Mit Rücksicht darauf, daß wir die p_r^s als große Zahlen betrachten können, haben wir

$$\lg W = \sum_s A^s \lg A^s - \sum_s\sum_r p_r^s \lg p_r^s,$$

wobei

$$A^s = \sum_r p_r^s.$$

Dieser Ausdruck soll ein Maximum sein unter der Nebenbedingung

$$E = \sum_s N^s h \nu^s; \quad N^s = \sum_r r p_r^s.$$

Die Durchführung der Variation liefert die Bedingungen

$$\sum_s \sum_r \delta p_r^s (1 + \lg p_r^s) = 0, \qquad \sum_s \delta N^s h \nu^s = 0$$

$$\sum_r \delta p_r^s = 0 \qquad \delta N^s = \sum_r r \delta p_r^s.$$

Hieraus folgt

$$\sum_s \sum_r \delta p_r^s (1 + \lg p_r^s + \lambda^s) + \frac{1}{\beta} \sum_s h \nu^s \sum_r r \delta p_r^s = 0.$$

Daraus folgt zunächst

$$p_r^s = B^s e^{-\frac{r h \nu^s}{\beta}}.$$

Da aber

$$A^s = \sum_r B^s e^{-\frac{r h \nu^s}{\beta}} = B^s \left(1 - e^{-\frac{h \nu^s}{\beta}}\right)^{-1},$$

so ist

$$B_s = A^s \left(1 - e^{-\frac{h \nu^s}{\beta}}\right).$$

Ferner hat man

$$N^s = \sum_r r p_r^s = \sum_r r A^s \left(1 - e^{-\frac{h \nu^s}{\beta}}\right) e^{-\frac{r h \nu^s}{\beta}}$$

$$= \frac{A^s e^{-\frac{h \nu^s}{\beta}}}{1 - e^{-\frac{h \nu^s}{\beta}}}.$$

Mit Rücksicht auf den oben gefundenen Wert von A^s ist also

$$E = \sum_s \frac{8 \pi h \nu^{s^3} d \nu^s}{c^3} V \frac{e^{-\frac{h \nu^s}{\beta}}}{1 - e^{-\frac{h \nu^s}{\beta}}}.$$

Mit Benutzung der bisherigen Resultate findet man ferner

$$S = k\left[\frac{E}{\beta} - \sum_s A^s \lg\left(1 - e^{\frac{h\nu^s}{\beta}}\right)\right],$$

woraus mit Rücksicht darauf, daß $\frac{\partial S}{\partial E} = \frac{1}{T}$, folgt, daß $\beta = kT$. Setzt man dies in obige Gleichung für E ein, so erhält man

$$E = \sum_s \frac{8\pi h\nu^{s^3}}{c^3} V \frac{1}{e^{\frac{h\nu^s}{kT}} - 1} d\nu^s,$$

welche Gleichung Plancks Formel äquivalent ist.

(Übersetzt von A. Einstein.)

Anmerkung des Übersetzers. Boses Ableitung der Planckschen Formel bedeutet nach meiner Meinung einen wichtigen Fortschritt. Die hier benutzte Methode liefert auch die Quantentheorie des idealen Gases, wie ich an anderer Stelle ausführen will.

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Planck's Law and the Light-Quantum Hypothesis

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The phase space of a light-quantum in a given volume is divided up in 'cells' of size h^3. The number of possible distributions of the light-quanta of a macroscopically defined radiation among the cells gives the entropy and with that all the thermodynamic properties of the radiation.

Planck's formula for the distribution of energy in the radiation of a blackbody forms the starting point for the quantum theory which has been developed in the last 20 years and has been very fruitful in all parts of physics. Since its publication in 1901 many methods for the derivation of this law has been proposed. It is recognized that the fundamental assumptions of the quantum theory are incompatible with the laws of classical electrodynamics. All derivations till now use the relation

$$\varrho_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} E,$$

that is, the relation between the density of radiation and the average energy of an oscillator, and they make assumptions about the number of degrees of freedom of the ether, which enters the above equation (the first factor on the right hand side). This factor could, however, be derived only from the classical theory. This is an unsatisfactory feature in all derivations, and it is not surprising that efforts are made again and again to give a derivation free from this logical flaw.

Einstein has given a remarkably elegant derivation. He has recognized the logical flaw in all previous derivations and has tried to deduce the formula independently of classical theory. Starting from very simple assumptions about the energy exchange between molecules and the radiation field, he finds the relation

$$\varrho_\nu = \frac{\alpha_{mn}}{e^{\frac{\varepsilon_m - \varepsilon_n}{kT}} - 1}.$$

In order to make this formula agree with that of Planck he has to use Wien's displacement law and Bohr's correspondence principle. Wien's law is based on classical theory and the correspondence principle assumes that the quantum theory agrees with the classical theory in certain limiting cases.

In all cases it appears to me that the derivations are not sufficiently justified from a logical point of view. On the other hand, the light-quantum hypothesis combined with statistical mechanics (as adapted by Planck to comform to the requirements of quantum theory) appears sufficient for the deduction of the law independent of classical theory. In the following I shall sketch the method briefly.

Let the radiation be enclosed in a volume V and its total energy be E. Suppose there are different types of quanta each having number N_s and energy $h\nu_s$ ($s = 0$ to $s = \infty$). The total energy is then

$$E = \sum_s N_s h\nu_s = V\int \rho_\nu \, d\nu. \tag{1}$$

The solution of the problem then requires the determination of of N_s which in turn determine ρ_ν. If we can give the probability for each distribution characterized by arbitrary values of N_s, then the solution is determined by the condition that this probability is a maximum subject to the subsidiary condition (1). We now want to find this probability.

The quantum has a momentum of magnitude $\frac{h\nu_s}{c}$ in the direction of its motion. The instantaneous state of the quantum is characterised by its coordinates x, y, z and the corresponding momenta p_x, p_y, p_z. These six quantities can be considered to be the coordinates of a point in a six dimensional space, where we have the relation

$$p_x^2 + p_y^2 + p_z^2 = \frac{h^2\nu^2}{c^2},$$

by virtue of which the point is forced to lie on the surface of a cylinder determined by the frequency of the quantum. The phase space belonging to the frequency interval $d\nu_s$ is.

$$\int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = V \,.\, 4\pi \left(\frac{h\nu}{c}\right)^2 \frac{h\, d\nu}{c} = 4\pi \frac{h^3\nu^2}{c^3} V d\nu.$$

If we divide the total phase space volume in cells of size h^3, then $4\pi V \frac{\nu^2}{c^3} d\nu$ cells will belong to the frequency interval $d\nu$. Nothing definite can be said about the method of this division. In any case, the total number of cells must be regarded as the number of possible arrangements of a quantum in the given volume. It seems, however, appropriate to multiply this number once again by 2 in order to take into account the fact of polarization, so that we obtain $8\pi V \frac{\nu^2 \, d\nu}{c^3}$ as the number of cells belonging to $d\nu$.

Now it is easy to calculate the thermodynamic probability (macroscopically defined) of a state. Let N^s be the number of quanta belonging to the frequency range $d\nu^s$. In how many ways can these be distributed among the cells belonging to $d\nu^s$? Let p_0^s be the number of empty cells, p_1^s those which contain one quantum, p_2^s those which contain two quanta and so on. The number of possible distributions is then,

$$\frac{A^s!}{p_0^s!\,p_1^s!\ldots}, \quad \text{where} \quad A^s = \frac{8\pi\nu^2}{c^3}\,d\nu^s,$$

and

$$N^s = 0\cdot p_0^s + 1\cdot p_1^s + 2p_2^s\ldots$$

is the number of quanta belong to the range $d\nu^s$.

The probability of the state defined by all p_r^s is clearly

$$\prod_s \frac{A^s!}{p_0^s!\,p_1^s!\ldots}.$$

Taking into account that we can consider p_r^s to be large numbers, we have

$$\lg W = \sum_s A^s \lg A^s - \sum_s \sum_r p_r^s \lg p_r^s,$$

where

$$A^s = \sum_r p_r^s.$$

This expression must be a maximum with the constraints

$$E = \sum_s N^s h\nu^s; \quad N^s = \sum_r r p_r^s.$$

Carrying out the variation, we obtain the conditions

$$\sum_s \sum_r \delta p_r^s (1 + \lg p_r^s) = 0, \qquad \sum_s \delta N^s h\nu^s = 0$$

$$\sum_r \delta p_r^s = 0 \qquad \delta N^s = \sum_r r\,\delta p_r^s.$$

From this it follows that

$$\sum_s \sum_r \delta p_r^s (1 + \lg p_r^s + \lambda^s) + \frac{1}{\beta} \sum_s h\nu^s \sum_r r\,\delta p_r^s = 0.$$

Next it follows from this that

$$p_r^s = B^s e^{-\frac{rh\nu^s}{\beta}}.$$

But since

$$A^s = \sum_r B^s e^{-\frac{rh\nu^s}{\beta}} = B^s\left(1 - e^{-\frac{h\nu^s}{\beta}}\right)^{-1},$$

therefore

$$B_s = A^s\left(1 - e^{-\frac{h\nu^s}{\beta}}\right).$$

We further have

$$N^s = \sum_r r p_r^s = \sum_r r A^s\left(1 - e^{-\frac{h\nu^s}{\beta}}\right) e^{-\frac{rh\nu^s}{\beta}}$$

$$= \frac{A^s e^{-\frac{h\nu^s}{\beta}}}{1 - e^{-\frac{h\nu^s}{\beta}}}.$$

Taking into account the value of A^s found above, we get

$$E = \sum_s \frac{8\pi h {\nu^s}^3 d\nu^s}{c^3} V \frac{e^{-\frac{h\nu^s}{\beta}}}{1 - e^{-\frac{h\nu^s}{\beta}}}.$$

Using the results found so far, one further finds

$$S = k\left[\frac{E}{\beta} - \sum_s A^s \lg\left(1 - e^{\frac{h\nu^s}{\beta}}\right)\right],$$

Whence, using the relation $\frac{\partial S}{\partial E} = \frac{1}{T}$, if follows that $\beta = kT$. Substituting this in the equation for E, we get

$$E = \sum_s \frac{8\pi h {\nu^s}^3}{c^3} V \frac{1}{e^{\frac{h\nu^s}{kT}} - 1} d\nu^s,$$

which is the same as Planck's formula.

(Translated by A. Einstein)

Translator's remarks :

In my opinion Bose's derivation signifies an important advance. The method used here gives the quantum theory of an ideal gas as I will work out elsewhere.

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Wärmegleichgewicht im Strahlungsfeld bei Anwesenheit von Materie.

Von **S. N. Bose** in Ramna (Indien).

(*Eingegangen am* 7, *Juli* 1924)

Die Wahrscheinlichkeit eines beliebigen Zustandes eines aus Strahlung und Molekeln bestehenden Systems wird berechnet, und es werden hieraus die Bedingungen fur das statistische Gleichgawicht abgeleitet. Im Anschluß hieran werden neue Ausdrücke für die statistische Wahrscheinlichkeit der Elementarvorgänge vorgeschlagen, welche der Wechselwirkung zwischen Strahlung and Materie entsprechen.

Debye[1] hat gezeigt, daß das Plancksche Gesetz mit Hilfe der statistischen Mechanik abgeleitet werden kann. Seine Ableitung ist jedoch insofern nicht völlig unbhängig von der klassischen Elektrodynamik, als er Gebrauch macht von dem Begriff der Eigenschwingungen des Äthers und annimmt, daß hinsichtlich der Energie das Spektralgebiet zwischen ν und $\nu+d\nu$ ersetzt werden kann durch $\frac{8\pi\nu^2}{c^3} V\,d\nu$ Resonatoren, deren Energie nur Vielfache von $h\nu$ betragen kann. Man kann aber zeigen, daß die Ableitung so abgeändert werden kann, daß man überhaupt keine Anleihen bei der klassischen Theorie *zu* machen braucht. $\frac{8\pi\nu^2}{c^3} V\,d\nu$ kann gedeutet werden als die Anzahl Elementargebiete des schesdimensionalen Phasenraumes für die Quanten. Die weitere Rechnung bleibt im wesentlishen dieselbe.

Einstein hat sich einer anderen Methode bedient. Er. betrachtet die Wechselwirkung zwischen den materiellen Telichen und dem Strahlungsfeld. Kennt man die Energieverteilung untr den materiellen Teilchen, so kann man das Gesetz der schwarzen Strahlung ermitteln aus der Bedingung der Stationarität dieser Verteilung beim Austausch von Energie zwischen Materie und Strahlung. Dieser Austausch sit weiterhin eine Folge gewisser Elementarprozesse. Eine geeignete Formulierung der Eigenschaften und Wahrscheinlichkeit des Elementar prozesses ermoglicht es, das Plancksche Gesetz abzuleiten, wenn man die Energieverteilung unter den Teilchen als gegeben ansieht. In seiner ersten Arbeit[2] sind die materiellen Gebilde Bohrsche Atome, die nur einer diskreten Reihe stationärer Zustände fähig sind. Energieaustausch findet statt infolge von Emissionsund Absroptionsprozessen und gleichzeitigen Übergängen der Atome von einem Zustand zu einem anderen. Er hat dargetan, daß das Plancksche Gesetz sich ergibt, wenn den Emissions- und Absorptionsprozessen gewisse Wahrscheinlichkeiten zugeschrieben werden. Neuerdings hat aber das Problem

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(1) Debye, *Ann. d. Phys.* **33**, 1427, 1910.

(2) Einstein, *Phys. ZS.* **18**, 121, 1917.

des thermischen Gleichgewichts zwischen dem Strahlungsfeld und freien Elektronen Wichtigkeit erlangst durch die Arbeiten von Debye(3) Compton(4) und anderen, die sich mit der Zerstreuung der Strahlung durch Elektronen befassen, Pauli(5) hat dieses Problem in einer interessanten Arbeit diskutiert. Als Elementarproze*ß* betrachtet er die Streuung, durch die ein Quant, das in einer bestimmten Richtung fliegt und innerhalb eines bestimmten Spektralintervalls.liegt, sich in ein anderes Quant eines anderen Spektralintervalls und von verschiedener Richtung verwandelt. Gleichzeitig andert ein Elektron, das sich mit einer gewissen Geschwindigkeit in einer bestimmten Richtung bewegt Gro*ß*e und Richtung seiner Geschwindigkeit. Für alle diese Prozesse gilt das Erhaltungsgesetz von Impuls und Energie. Pauli hat gezeigt, da*ß*, wenn die Wahrscheinlichkeit dieses Elementarprozesses $(A\rho+B\rho\rho')dt$ ist, wo ρ und ρ' die Strahlungsdichten der Frequenzen ν und ν', A und B unabhängig von q and q' aber abhänging von der Art des betrachteten Sto*ß*es sind, dieses Wahrscheilichkeitsgesetz zum Planckschen Gesetz führt, falls für das Elektronengas das Maxwellsche Verteilungsgesetz gültig ist. Einstein unde Ehrenfest haben das Paulische Ergebnis noch verallgemeinert(6). Sie betrachten einen Proze*ß*, bei dem ein Molekül order ein Elektron Quanten $h\nu_1$, $h\nu_2$ usw. absorbiert und gleichzeitig $h\nu'_1$, $h\nu'_2$ ...emittiert. Es wird angenommen, da*ß* die emittierten und absorbierten Quanten bestimmte Richtungen haben. Zugleich andert sich auch Gro*ß*e und Richtung der Geschwindigkeit des betreffenden Moleküls. Energie und Impuls bleiben aber erhalten. Das Wahrscheinlichkeitsgesetz von Pauli wird folgenderma *ß* on verallgemeinert :

$$dW_1 = \Pi b_1\rho_1\Pi(a'_1+b'_1\,\rho'_1)dt,$$

$$dW_2 = \Pi(a_1+b_1\rho_1)\Pi(b'_1\,\rho'_1)dt.$$

für den direkten und inverson Proze*ß*.

Er zeigt, da*ß* Gleichgewicht vorhanden ist, wenn gewisse Beziehungen zwischen den Koeffizienten bestehen. Dies ist sowohl eine Verallgemeinerung seines früheren Gesetzes fur Bohrsche Atom wide des Gesetzes von Pauli

Das Problem des Warmegleichgewichts in einem Strahlungsfeld in Anwesenheit von materiellen Gebilden kann aber nach den Methoden der statistischen Mechanik behandelt werden, unabhängig von jeder besonderen Hypothese über den Mechanismus der Elementarprozesse, auf denen der Energieaustausch beruht. Wir gelangen so zu einer allgemeinen Beziehung, die güetig ist für alle speziellen Hypothesen über die Elementarprozesse und ihre Wahrscheinlichkeiten. Wenn es moglich ist, die thermodynamische Wahrescheinlichkeit für irgend eine spezielle Verteilung der

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(3) Debye, *Phys. ZS.* **24**, 161, 1923.

(4) Compton, *Phys. Rev.* **21**, 483, 1923.

(5) Pauli, *ZS. f. Phys.* **18**, 272, 1923.

(6) Einstein und Ehrenfest, *ZS. f. Phys.* 19, 301, 1923.

Quanten im Strahlungsfeld und die Wahrscheinlichkeit für irgend eine willkürliche Verteilung der Enerige unter den materiellen Teilchen niederzuschreiben, dann ist die thermodynamische Wahrscheinlichkeit des größeren Systems, das Materie unc Strahlung umfaßt, einfach das Produkt der bieden Wahrscheinlichketien. Die Gleichgewichtsbedingung lautet einfach, daß die resultierende Wahrscheinlichkeit ein Maximum sein muß. Das Gleichgewichts-problem kann somit in einer ganza ganz anderen Weise betrachtet werden. Statt geeignete Hypothesen uber Elementarprozesse, die das Plancksche Gesetz ergeben, aufzustellen, versuchen wir, die Relation abzuleiten, zu der alle Hypothesen fuhren mussen, wenn das Plancksche Gesetz fur das Strahlungsfeld und das Maxwellsche Gesetz für die materiellen Teilchen gelten sollen. Die Beziehung, die wir suchen, ist offenbar aquivalent mit der der zuvor aus der Bedingung, daß die thermodynamischche Wahrscheinlichkeit für das größ ere System ein Maximum ist, aabgeleiteten. Um das Problem in der angedeuteten Weise zu lüsen, mussen wir den Ausdruck für die thermodynamische Wahrscheinlichkeit für die Strahlung und für die materiellen Teilchen haben.

Thermodynamische Wahrscheinlichkeit für das Strahlungsfeld. Sie wurde in einer fruherent Arbeit(7) abgeleitet. Wenn die Zahl der Quanten im Spektralbereich ν und $\nu+d\nu$ gleich $N\,d\nu$ ist, so ist sie :

$$W = \prod_{\nu} \frac{(A_\nu+N_\nu d\nu)!}{A_\nu!\,N_\nu d\nu\,!},$$

wo $A_\nu = \frac{8\pi\nu^2}{c^3} V\,d\nu$ bedeutet. Es ist leicht zu sehen, daß dieses Wahrscheinlichkeitsgesetz zum Planckschen Gesetz führt, wenn wir die Bedingung aufstellen, daß W bei gegebener Energie ein Mximum sein muß.

Thermodynamische Wahrscheinlichkeit für materielle Teilchen. Man kann sic unschwer finden. Wir wollen unsere Voraussetzungen ein wenig verallgemeinern, um sowohl den Fall Bohrscher Atome mit diskreten Energieniveaus als auch den Fall der Verteilung der Translationsenergie bei materiellen Teilchen(8) zu umfassen. Wir nehmen an, der Phasenraum sie in Elementargebiete geteilt. Zu jeder Elementarzelle gehort eine bestimmte Zahl g, die die Wahrscheinlichkeit angibt, *daß* irgend ein Teilchen sich darin befindet. Die g sind im algemeinen einander gleich, ausgenommen den Fall Bohrscher Atome. Die thermodynamische Wahrscheinlichkeit irgend einer willkurlichen Verteillung n_1, n_2 use uber die verschiedenen Zellen ist :

$$\frac{N!\,g_1^{n_1} g_2^{n_2}\ \dots}{n_1!\,n_2!\ \dots}.$$

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(7) Der Verfasser, erscheint im Phil. Mag. Siche auch Debye, *Ann. d. Phys.* **33**, 1427, 1910.

(8) Bohr, *ZS. f. Phys.* **18**, 117, 1923.

Die Bedingung des Wahrscheinlichkeitsmaximums bei gegebener Gesamtenergie und Zahl der Teilchen ist :

$$n_r = C g_r \, e^{-\frac{E}{KT}} \, .$$

Jetzt sind wir in der Lage, die thermodynamische Wahrscheinlichkeit für das gesamte System zu berechnen. Sie ist offenbar :

$$W = \prod_s \frac{A_s + N_s!}{A_s! N_s!} \prod_T \frac{g_T^{n_T} N!}{n_T!} \, ,$$

wo

$$\Sigma N_s h\nu_s + \Sigma n_T E_T = E$$

und

$$\Sigma n_T = N.$$

Beim Gleichgewicht ist *W* ein Maximum, wobei die Gesamtenergie und die Zahl der Teilchen als gegeben anzusehen ist.

Wir betrachten den Elementarprozeβ :

n_r verwandelt sich in $n_r - 1$ und n_s in $n_s + 1$; N_{ν_1}, N_{ν_2} ... vermindert sich um 1 und $N_{\nu_1'}$, $N_{\nu_2'}$ ... wächst um 1.

Die Bedingung der Stationaritat von *W* verlangt, daβ *W* sich dabei nicht ändert.

Der betrachtete Elementarproze kann aufgefaβt werden als der Übergang eines Teilchens vom *r*-ten zum *s*-ten Zustand. Die Veränderung, die das Strahlungsfeld dabei erleidet, kann angesehen werden als das Ergebnis einer Zerstreuung nach einem Zusammenstoβ.

Die gesuchte Bedingung lantet :

$$\frac{n_r}{g_r} \prod_\nu \frac{N_{\nu_r}}{N_{\nu_r} + A_{\nu_r}} = \frac{n_s}{g_s} \prod_{\nu'} \frac{N_{\nu'}}{N_{\nu'} + A_{\nu'}} \qquad \dots \quad \text{(I)}$$

wo

$$\Sigma h\nu' - \Sigma h\nu + E_s - E_r = 0.$$

Diese Gleichung ist offenbar äquivalent mit

$$\frac{n_r}{g_r} \prod \frac{b_1 \rho_{\nu_1}}{a_1 + b_1 \rho_\nu} = \frac{n_s}{g_s} \prod \frac{b'_1 \rho_{\nu'}}{a'_1 + b'_1 \rho'_\nu} \, ,$$

wo

$$\frac{a_1}{b_1} = 8\pi \frac{\nu^2}{c^3} h\nu, \; \frac{a_1'}{b_1'} = 8\pi \frac{\nu'^2}{c^3} h\nu'.$$

Dies kann auch geschrieben werden :

$$n_r \Pi b_1 \rho_{\nu_1} \Pi (a_1' + b_1' e'_\nu) = n_s \Pi b_1' \rho_\nu^1 \Pi (a_1 + b_1 \rho_\nu),$$

wenn

$$g_r = g_s.$$

Das ist die verallegemeinerte Gleichung, die fur Elementarprozesse von der Art, wie sie Einstein und Ehrenfest[9] betrachetet haben, gilt. Nach der hier gegebenen Ableitung ergibt sich die Beziehung als eine Folgerung aus der Bedingung, daβ beim Gleichgewicht die Wahrscheinlichkeit ein Maximum sein muβ. Wenn aber der Mechanisms des Elementarprozesses uns gestattet, die Gleichung direkt niederzuchreiben fuhrt die Substitution des Wertes fur das Verhältnis $\frac{n_r}{n_s}$ zum Planckschen Gesetz.

Es is jedoch klar, daβ Gleichung (I) fundamentaler ist als die transformierte Form von Einstein. Die nachfolgende Betrachtung wird zeigen, daβ eine einfache Uberlegung den Wert des Wahrscheinlichkeitskoeffizienten in der gewünschten Form liefert.

Der Fall des Bohrschen Atoms.

Das ist der Elementarprozeβ, den Einstein[10] in seiner fruheren Arbeit betrachtet hat.

Die fundamentale Gleichung (I) reduziert sich hier auf

$$\frac{n_r}{g_r} \frac{N_\nu}{A_\nu + N_\nu} = \frac{n_s}{g_s}.$$

Einsteins ursprüngliche Gleichung lautet :

$$n_r b_1 \rho_\nu dt = n_s(a_1' + b_1' \rho_\nu)dt \qquad \dots \quad (1)$$

oder

$$\rho_\nu = \frac{a_1' n_s}{n_r b_1 - n_s b_1'}.$$

Nimmt man ferner an, daβ

$$g_r b_1 = g_s b_1' \quad \text{und} \quad \frac{a_1'}{b_1'} = 8\pi \frac{h\nu^3}{c^3},$$

so folgt das Plancksche Gesetz.

Um seine Gleichung (I) abzuleiten, nimmt er an, daβ Atome von höheren Energieniveaus zu Zuständen niederer Energie auf zwei Wegen übergehen konnen :

1. Eine Art von spontanem Übergang, wie bein einem radioactiven Prozeβ, dessen Wahrscheinlickkeit unäbhangig vom Zustand des Strahlungsfeldes ist.

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(9) Einstein und Ehrenfest, a. a. O.

(10) Einstein, *Phys. ZS.* **18**, 121, 1917.

2. Ein durch die charakteristische Strahlung induzierter Übergang, dessen Wahrscheinlichkeit vom Zustand des Feldes abhängt.

Der Übergang von einem niederen zu einem höheren Energieniveau erfolgt durch Absorption, deren Wahrscheinlichkeit vom Felde abhängt.

Weiterhin werden gewisse Benziehungen zwischen den Koeffizienten postuliert, damit das Plancksche Gesetz herauskommt. Diese Beziehungen konnen nicht auf einfache Weise begründet werden.

Das gleiche Problem kann auf die folgende Weise behandelt werden, bei der viele von den vorkommenden willkürlichen Annahmen beseitigt werden.

Wir setzen auch voraus, daß der Übergang von niederen zu höheren Niveaus stets durch Absroption von Quanten einer charakteristischen Frequenz erfolgt. A. Der Übergang von höheren zu niederen Niveaus ist eine Art von spontaner Änderung, dessen Wahrscheinlichkeit ünabhangig vom Strahlungsfeld ist. Die zweite Hypothese von Einstein (negative Einstrahlung) ist nicht erforderlich. B. Wir wollen versuchen, den Wahrscheinlichkeitskoeffizienten für Übergang A zu berechnen.

Die Zahl der Elementargebiete, die den in Betracht kommenden Quanten entsprechen, ist A_ν. Die Gesamtzahl der vorhandenen Quanten ist $N_\nu d\nu$. Diese sind aber nicht gleichförming über den Phasenraum verteilt. Sei p_0 die Zahl der leeren Zellen, p_1 die *mit* 1 Quant, p_2 die *mit* 2 Quanten usw. besetzten. Damit die Strahlung und das materielle Teilchen in Wechselwirkung treten können, muß is dem Elementargebiet sein, das von einem Quant besetzt ist. Das besondere Elementargebiet, in dem sich das betrachtete Molekül befindet, wird, wenn wir es lange genug beobachten, alle moglichen Zustände durchlaufen : d.h. manchmal ist es leer, manchmal wird 1 Quant vorhanden sein, manchmal 2 usw. Die Dauer dieser Ereignisse wird Schleißlich proportional zu den Gleichgewichtswerten von $p_0, p_1 \ldots$ Wenn r Quanten und ein Teilchen zugleich vorhanden sind, dann gibt es entweder einen Energieaustausch oder gar nichts. Somit sind $r+1$ verschiedene Ereignisse möglich, nämlich Energieaustausch von 1 Quant, 2 Quanten usw. oder kein Austausch

Daher ist die Zahl der möglichen Fälle :

$$p_0+2p_1+3p_2+\ldots = A_s+N_s d\nu_s = \Sigma(r+1)p_r.$$

Die Zahl der Fälle, in denen Wechselwirkung eintritt, ist :

$$p_1+2p_2+\ldots = N_s d\nu_s = \Sigma r p_r.$$

Somit ist die Wahrscheinlichkeit einer Wechselwirkung :

$$\frac{\Sigma\, r p_r}{\Sigma(r+1)p_r} = \frac{N_s d\nu_s}{A_s+N_s d\nu_s}.$$

Nun mussen wir die sgezielle Art der Wechselwirkung in Rucksicht ziehen, d.h. die Absorption, die zu unterscheiden ist von Zerstreuung. Die Wahrscheinlichkeit hierfür sie β.

Die Wahrscheinlichkeit der Absorption ist :

$$\frac{\beta \Sigma r p_r}{\Sigma (r+1) p_r} = \beta \frac{N_\nu d\nu}{A_\nu + N_\nu d\nu} \ .$$

Der Wahrscheinlichkeitskoeffizient für einen Übergang vom Typus B ist offenbar irgend eine Konstante, die den Übergang von dem speziellen höheren Zustand zu dem speziellen niederen Zustand bestimmt. Diese Konstante sei α.

Die Gleichgewichtsbedingung kann dann geschrieben werden :

$$n_r \beta \frac{N_\nu d\nu}{A_\nu + N_\nu d\nu} = \alpha n_s$$

was mit der fundamentalen Gleichung (I) übereinstimmt, wenn wir annehmen, daß

$$g_r \beta = g_s \alpha.$$

Es ist klar, daß die Substitution von $\frac{n_r}{n_s}$ zum Planckschen Gesetz führt.

Der Fall von Pauli.

Er ist einfach zu behandeln. Die Wahrscheinlichkeit einer Wechselwirkung ist wire vorhin $\frac{\Sigma r p_r}{\Sigma (r+1) p_r} \cdot \beta^s_{s'}$ sei der Wahrscheinlichkeitskoeffizient für Zerstreuung von ν_s in $\nu_{s'}$ falls eine Wechselwirkung statthat. Für die inverse Transformation seien die entsprechenden Koeffizienten $\frac{\Sigma r p'_r}{\Sigma (r+1) p'_r}$ und $\beta^{s'}_s$.

Beim Gleichgewicht ist also :

$$n_r \beta^s_{s'} \frac{N_s d\nu_s}{A_s + N_s d\nu_s} = n_s \beta^{s'}_s \frac{N_{s'} d\nu_{s'}}{A_{s'} + N_{s'} d\nu_{s'}} \ .$$

Nehmen wir ferner an, daß $\beta^s_{s'} = \beta^{s'}_s$ so erhalten wir die fundamentale Gleichung (I).

Der Fall von Einstein und Ehrenfest.

Die Wahrscheinlichkeit einer gleichzeitigen Wechselwirkung ist das Produkt der Einzelwahrscheinlichkeiten, also gleich :

$$\Pi \frac{N_s d\nu_s}{A_s + N_s \, d\nu_s} \ .$$

Analog sei $\beta^{s_1\ s_2\ \cdots}_{s'_1\ s'_2\ \cdots}$ die Wahrscheinlichkeit für den speziellen Akt der Zerstreuung, und ein ähnlicher Ausdruck gelte für die inverse Streuung, woraus sich das Übrige leicht ergibt.

Bemerkungen. Aus unserer Rechung geht hervor, daß die Wahrscheinlichkeit einer Wechselwirkung eines Teilchens mit einem Quantum in einem Strahlungsfeld $\frac{N_\nu d\nu}{A_\nu + N_\nu d\nu}$ nicht einfach proportional zur Zahl der vorhandenen Quanten ist, wie man auf den ersten Blick vermuten könnte. Es ist leicht zu sehen, daß diese Annahme zum Wienschen Gesetz führen würde, wie Pauli erkannt hat⁽¹¹⁾. Pauli sah sich deshalb genotigt, eine andere Form für den Wahrscheinlichkeitskoeffizienten nzuunehmen. Die von Pauli angenommene und von Einstein und Ehrenfest verallgemeinerte Form erscheint aber als ganz willkurliche Hypothese, da man sich kein einfaches Bild von dem Zustandekommen eines solchen Ausdrucks machen kann. Die andere hier vorgeschlagene Form ist siemlich einfach und kann auf Grund elementarer Überlegungen gerechtfertigt werden. Es wird auch die Notwendigkeit vermieden, Felationen zwischen den Koeffizienten selbst annehman zu müssen. Bei der Ableitung des Wahrscheinlichkeitskoeffizienten für die Wechselwirkung (oder Kopplung, wie Bohr sagt) wurde angenommen, daß selbst bei einem Zusammenstoß der Fall, daß keine Wechselwirkung eintritt, gerade so wahrscheinlich ist, wie der Fal irgend einer spezielen Kopplung Diese Annahme ist ein fundamentaler Punkt in der hier gegebenen Ableitung. Nach der klassischen Theorie würde man erwarten, daß irgend eine Wechselwirkung eintritt, wenn ein Quant mit einem materiellen Teilchen zusammentrifft. Eshandelt sich also um eine principielle Abweichung von der klassischen Theoire. Diese Hypothese ist aber (wie mir scheint) ganz ähnlich wie die Hypothese, die man gewohnlich über die Stabilität der stationaren Zustände macht. Die klassische Theorie ließe uns erwarten, daß keine stationären Zustände moglich sind, und daß stets Wechselwirkung oder Kipplung (Bohr) zwischen Äther und erregtem Atom und damit verbundene Strahlung vorhanden ist. Auf der anderen Seite mussen wir, um die spektrale Emission zu erklaren, den Fall als moglich annehmen, daß keine Wechselwirkung (Emission) eintritt. So kommen wir dauz, wegen der Persistenz der stationaren Zustande jedem beliebigen stationären Zustand einen Wahrscheinlickkeitskoeffizienten oder eine mittlere Lebensdaur zususchreiben.

Die oben benutzte Hypothese, daß aich bei einem Zuszmmenstoß keine Wechselwirkung eintreten kann, ist ganz analog der Annahme der Stabilität der stationären Zustände, die so grundlegend für die Bohrsche Theorie der Linienemission ist, und kann auf die gleiche Ursache zurückgefuhrt werden : Die den materiellen Teilchen innewohnende Wahrscheinlichkeit der Persistenz ihrer stationären Zustände. In

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(11) Pauli, *Z.S f. Phys.*, a a 0).

diesem Zusammenhang ist es interessant, zu bemerken, daß wir beim analogen Fall des Zusammenstoßes eines Elektrons und eines Atoms experimentell Fälle nachweisen konnen, wo bei Zusammenstoßen das Elektron durch das Atom hindurchgeht, ohne die intraatomare Bewegung oder seine eigene zu verandern.

Die Beziehung $g_r\beta = g_s\alpha$.

Die Koeffizienten, oder wie Bohr[12] sie nennt, Gwichte g werden eingefuhrt, wenn Valenzelektron im Atom eine entartet bedingt periodische Bewegung ausfuhrt. Diese numerischen Koeffizienten geben nach Bohr einfach an, auf wieviel verschiedene Weisen man von einer benachbarten nichtentarteten Bewegung zu der speziellen entarteten Bewegung als Grenzfall gelangen kann. Nehmen wir diese Koeffizienten einfach als proportional zur Zahl der Übergangsmöglichkeiten von einem bestimmten zu dem betrachteten Zustand an, so ist die obige Beziehung fast selbstverständlich. Es gibt g_s Wege, auf denen ein Übergang von irgend einem r-ten zu dem betrachteten s-ten Zustand durch Absorption möglich ist. Analog gibt es g_r Übergänge zu niederen Zuständen infolge von Emission. Der Wahrscheinlichkeitskoeffizient hat dann den Faktor g_s der Wahrscheinlichkeitskoeffizient der Emission den Faktor g_r. Wenn wir ferner annehmen, daß die Wahrschelichkeit einer Transformation durch Strahlung auf einem bestimmten Wege die gleiche ist wie der Wahrscheinlichkeitskoeffizient der Emission, wenn das Atom in bestimmter Weise zu niederen Zuständen übergeht, so ergibt sich sofort die fragliche Relation[13].

Manindra Physikal Laboratorium, Dacca-Universität, 14 Juni 1924.

Ich halte Boses Hypothese uber die Wahrscheinlichkeit der Strahlungselementarvorgange aus folgenden Grunden fur nicht zutreffend.

Für das statistische Gleichgewicht zwischen einem Bohrschen Zustande zu einem anderen gilt, wie Bose dargelegthat, die Beziehung

$$\frac{n_r}{g_r}\,\frac{N_\nu}{A_\nu+N_\nu}=\frac{n_s}{g_s}$$

Daraus folgt, daß die Wahrscheinlichkeiten für die Übergange $r \to s$ und $s \to r$ der linken bzw. rechten Seite dieser Gleichung proportional sein müssen. Die Übergangswahrscheinlichkeiten für. ein Molekül müssen sich also (wenn wir der Einfachheit halber die statistichen Gewichte beider Zustände gleich 1 setzen) wie $\frac{A_\nu}{A_\nu+N_\nu}$: 1 verhalten. Mehr kann aus der Kenntnis des thermodynamichen Gleichgewichts

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(12) Bohr, *Z.S f. Phys.*, a.a.O.

(13) Vgl. in diesem Zusammenhang P. Hertz, *Repert. d. Phys.*, statistische Mechanik, 1. *Bd.*, Teil 2, S. 549.

nicht entonommen werden. Nach der von mir auftestellten Hypothese sollen diese Wahrscheinlichkeiten N_ν (d.h. der Strahlungsdichte) bzw. $A_\nu + N_\nu$ proportional sein, nach der Hypothese Boses $\frac{N_\nu}{A_\nu + N_\nu}$ *bzw.* 1.

Nach der letzteren Hypothese kann die aüßere Strahlung wohl einen Übergang von Zustande Z, kleinerer Energie nach dem Zustande Z_s größerer Energie bewirken, nicht aber umgekhrt einen Übergang von Z_s nach Z_r. Dies widrespricht aber dem mit Recht allgemein anerkannten Grundsatz, daß die klassische Theorie einen Grenzfall der Quantentheorie darstellen müsse. Nach letzterer kann nämlich ein Strahlungsfeld auf einen Resonator sowohl positive als auch negative Energie übertragen (je nach der Phase), und zwar beides gleich wahrscheinlich. Die Wahrscheinlichkeiten beider Übergange müssen also von der Strahlungsdichte, d.h. von N_ν abhängen, im Gegensatz zu Boses Hypothese. Inwiefern die Qunatentheorie die klassische zum Grenzfall hat, hat Planck in der letzten Auflage seines Buches über Strahlungstheorie ausführlich erötert.

Zweitens müßte nach Boses Hypothese ein kalter Körper ein von der Strahlungsdichte abhängiges (mit ihr abnehmendes) Absorptionsvermögen besitzen. Die Körper sollten in Kaltem Zustande "nicht-Wiensche" Strahlung schwächer absorbieren als weniger intensive aus dem Gültigkeitsbereich der Wienschen Strahlungsformel. Dies wäre bei ultraroter Strahlung heißer Lichtquellen gewiß schon entdeckt worden, wenn es sich so verhielte.

A. Einsten

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Thermal Equilibrium in the Radiation Field in the Presence of Matter

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The probability of an arbitrary state of a system consisting of radiation and molecules is calculated and from it the conditions for statistical equilibrium are derived. Further, new expressions for the statistical probability of elementary processes, appropriate for the interaction between radiation and matter, are proposed.

Debye[1] has shown that Planck's law can be derived using statistical mechanics. His derivation is, however, not completely independent of classical electrodynamics, because he uses the concept of normal modes of the ether and assumes that for calculating the energy the spectral range between ν and $\nu + d\nu$ can be replaced by $\frac{8\pi\nu^2}{c^3} V d\nu$ resonators whose energy can be only multiples of $h\nu$. One can however show that the derivation can be so modified that one does not have to borrow anything from the classical theory. $\frac{8\pi\nu^2}{c^3} V d\nu$ can be interpreted as the number of elementary cells in the six dimensional phase space of the quanta. The further calculations remain essentially unchanged.

Einstein has used another method. He considers the interaction between material particles and the radiation field. If one knows the energy distribution among the particles, then one can derive the law of black-body radiation from the condition of stationarity of this distribution when there is an exchange of energy between matter and radiation. This exchange is moreover a result of certain elementary processes. With a suitable formulation of properties and the probability of these elementary processes one can derive Planck's law, if the energy distribution among the particles is assumed to be known. In his first paper[2] the material particles are Bohr atoms which can have only a series of discrete stationary states. The energy exchange occurs because of emission and absorption processes and the simultaneous transitions of atoms from one state to another. He has shown that Planck's law is obtained when the emission and absorption processes have certain probabilities. Recently the problem of thermal equilibrium between radiation and and free electrons has acquired added importance through the studies of Debye[3], Compton[4] and others, which deal with the scattering of radiation off electrons. Pauli[5] has discussed this problem in an interesting paper. As an elementary process he considers the scattering of an electron by a quantum which moves in a definite direction and has a frequency in a given spectral range and is converted into another quantum with a different frequency and moving in a different direction. Simultaneously an electron which moves with a certain velocity in a given direction changes the magnitude and the direction of its velocity. The laws of conservation of energy and momentum hold for all these processes. Pauli has shown that if the probability of this elementary process is $(A\rho + B\rho\rho')\,d\nu$ where ρ and ρ' are radiation densities for frequencies ν and ν', A and B are independent of ρ and ρ' but depend on the nature of collision, this probability law leads to Planck's law, provided Maxwell's distribution is valid for the electron gas. Einstein and Ehrenfest[6] have further generalized Pauli's result. They consider a process in which a molecule or an electron absorbs quanta $h\nu_1$, $h\nu_2$, etc. and simultaneously emits $h\nu_1'$, $h\nu_2'$ ... It is assumed that the emitted and absorbed quanta have definite directions. The magnitude and the direction of the velocity of the molecule under consideration also change at the same time. Energy and momentum are however conserved. The probability law of Pauli is generalized as follows,

$$dW_1 = \Pi b_1 \varrho_1 \Pi (a_1' + b_1' \varrho_1')\,dt,$$
$$dW_2 = \Pi (a_1 + b_1 \varrho_1) \Pi (b_1' \varrho_1')\,dt,$$

for the direct and the inverse processes. They show that equilibrium is obtained if certain relations between the coefficients exist. This is a generalization of Einstein's earlier result for Bohr atoms and also of Pauli's result.

The problem of thermodynamic equilibrium of radiation in the presence of material particles can, however, be studied using the methods of statistical mechanics, independently of any special assumption about the mechanism of the elementary processes on which the energy exchange depends. In this way we obtain a general relation which is valid for all special assumptions about the elementary processes and their probabilities. If it is possible to write down the thermodynamic probability for any special distribution of quanta of radiation and the probability for any arbitrary energy distribution among the particles, then the thermodynamic probability for the bigger system containing matter and radiation is simply the product of the two probabilities. The condition of equilibrium is simply that the resulting probability must be a maximum. The problem of equilibrium can thus be looked at in a completely different way. We try to derive the relation to which all assumptions must lead when Planck's law for radiation and Maxwell's law for particles hold instead of making assumptions about elementary processes adopted to give Planck's law. The relation which we are looking for is clearly equivalent to the previous one which is derived from the condition that the thermodynamic probability for the bigger system is a maximum. In order to solve the problem in the way indicated, we must have the expressions for the thermodynamic probability for radiation and the material particles.

Thermodynamic probability for radiation :

This has been derived earlier[7]. If $N_\nu\,d\nu$ is the number of quanta in the frequency range ν and $\nu + d\nu$, then the probability is,

$$W = \prod_{\nu} \frac{(A_\nu + N_\nu d\nu)!}{A_\nu!\, N_\nu d\nu!},$$

where $A_\nu = \frac{8\pi\nu^2}{c^3} V d\nu$. It is easily seen that this probability law leads to Planck's law if we impose the condition that for a given energy W should be a maximum.

Thermodynamic probability for particles :

This can be easily found. We want to generalize our assumptions a little so that the case of the Bohr atom with discrete energy levels as well as the case of the distribution of translational energy of particles can be included. Let the phase space be divided in to cells. For every cell there is a probability g that a particle occupies it. The g's are in general equal except for the case of Bohr's atoms. The thermodynamic probability for any arbitrary distribution n_1, n_2 etc. among the different cell is,

$$\frac{N!\, g_1^{n_1} g_2^{n_2} \ldots}{n_1!\, n_2!}.$$

The condition that the probability be maximum for a given total energy and number of particles is,

$$n_r = C g_r c^{-\frac{E}{kT}}.$$

Now we are in a position to calculate the thermodynamic probability for the total system. It is clearly

$$W = \prod_{s} \frac{A_s + N_s!}{A_s!\, N_s!} \prod_{T} \frac{g_T^{n_T} N!}{n_T!},$$

where

$$\Sigma N_s h\nu_s + \Sigma n_T E_T = E$$

and

$$\Sigma n_T = N.$$

At equilibrium W is a maximum, the total energy and number of particles being given.

We consider the following elementary process : n_r changes to $n_r - 1$ and n_s to $n_s + 1$; N_{ν_1}, N_{ν_2}, ... reduce by 1 and $N_{\nu'_1}$, $N_{\nu'_2}$ increase by 1.

The condition of stationarity of W requires that W does not change on account of this process.

The elementary process under consideration can be considered as the transition of a particle from the rth to the sth state. The change which the radiation field undergoes can be looked upon as the result of a scattering after a collision. The required condition reads

$$\frac{n_r}{g_r}\underset{\nu}{\Pi}\frac{N_{\nu_r}}{N_{\nu_r}+A_{\nu_r}}=\frac{n_s}{g_s}\underset{\nu'}{\Pi}\frac{N_{\nu'}}{N_{\nu'}+A_{\nu'}}\,. \tag{I}$$

where

$$\sum h\nu' - \sum h\nu + E_s - E_r = 0.$$

The equation (I) is evidently equivalent to

$$\frac{n_r}{g_r}\Pi\frac{b_1\,\varrho_{\nu_1}}{a_1+b_1\,\varrho_\nu}=\frac{n_s}{g_s}\Pi\frac{b_1'\,\varrho_\nu'}{a_1'+b_1'\,\varrho_\nu'}.$$

where

$$\frac{a_1}{b_1}=8\pi\frac{\nu^2}{c^3}h\nu,\qquad \frac{a_1'}{b_1'}=8\pi\frac{\nu'^2}{c^3}h\nu'.$$

This can also be written as

$$n_r\,\Pi\,b_1\,\varrho_{\nu_1}\,\Pi\,(a_1'+b_1'\,\varrho_\nu')=n_s\,\Pi\,b_1'\,\varrho_\nu'\,\Pi\,(a_1+b_1\,\varrho_\nu),$$

if

$$g_r = g_s.$$

This is the generalized equation which holds for elementary processes of the type considered by Einstein and Ehrenfest[6]. According to the derivation given here the relation follows from the condition that for equilibrium the probability must be a maximum. However, if the mechanism of the elementary process allows us to write down the equation directly, the substitution of the value for the ratio $\frac{n_r}{n_s}$ gives Planck's law.

It is however clear that the equation (I) is more fundamental than the original one of Einstein. The following discussion will show that simple considerations give the value of the probability constants in their desired form.

The case of Bohr's atoms :

This is the elementary process which Einstein had considered in his earlier paper[2]. The fundamental equation (I) here reduces to

$$\frac{n_r}{g_r}\frac{N_\nu}{A_\nu+N_\nu}=\frac{n_s}{g_s}.$$

Einstein's original equation states

$$n_r\,b_1\,\varrho_\nu\,dt = n_s\,(a_1'+b_1'\,\varrho_\nu)\,dt$$

or

$$\varrho_\nu=\frac{a_1'\,n_s}{n_r\,b_1-n_s\,b_1'}\,.$$

If we assume further that

$$g_r\,b_1 = g_s\,b_1' \quad\text{and}\quad \frac{a_1'}{b_1'}=8\pi\frac{h\nu^3}{c^3},$$

then Planck's law follows :

To derive his equation (I), Einstein assumes that atoms in higher energy levels make transitions to lower energy levels in two ways :

1. A type of spontaneous transition as in a radioactive process, whose probability is independent of the state of the radiation field.
2. A transition induced by the characteristic radiation, whose probability depends on the state of the radiation field.

The transition from a lower to a higher energy level is the result of an absorption whose probability depends on the radiation field. Further, certain relations between the coefficients are postulated so that Planck's law follows. These relations cannot be justified in a simple way.

The same problem can be treated in the following way removing many of the arbitrary assumptions.

We also presume that the transition from a lower to a higher state takes place through an absorption of a quantum of characteristic frequency.

A. The transition from higher to lower levels is a spontaneous change, whose probability is independent of the field of radiation. The second assumption of Einstein (negative radiation) is not necessary.

B. We shall try to calculate the probability coefficients for the transition *A*.

The number of cells corresponding to the quanta under consideration is A_ν. The total number of quanta present is $N_\nu \, d\nu$. They are however not distributed evenly in phase space. Let p_o be the number of empty calls, p_1 with 1 quantum, p_2 with two quanta etc. In order that the radiation and a particle may interact, it must be in a cell occupied by a quantum. The particular cell which the molecule under consideration occupies will pass through all possible states if we observe it long enough, that is, sometimes it is empty, sometimes occupied by 1 quantum sometimes by 2 quanta etc. The length of these events will be finally proportional to the equilibrium values of p_o , p_1. When r quanta and a particle are together, then either an exchange of energy takes place or nothing happens. Therefore $r + 1$ different events are possible, namely energy exchange of 1 quantum, 2 quanta etc. or no exchange. Therefore the number of possible cases is

$$p_0 + 2\,p_1 + 3\,p_2 + \cdots = A_s + N_s\,d\,\nu_s = \Sigma\,(r+1)\,p_r.$$

The number of cases in which interactions occur is,

$$p_1 + 2\,p_2 + \cdots = N_s\,d\,\nu_s = \Sigma\,r\,p_r.$$

Consequently the probability of an interaction is,

$$\frac{\Sigma\,r\,p_r}{\Sigma\,(r+1)\,p_r} = \frac{N_s\,d\,\nu_s}{A_s + N_s\,d\,\nu_s}.$$

Now we must take into consideration the special nature of the interaction, that is, absorption which is to be distinguished from scattering. Let the probability for this be β. The probability for absorption is

$$\frac{\beta \sum r p_r}{\sum (r+1) p_r} = \beta \frac{N_\nu d\nu}{A_\nu + N_\nu d\nu}.$$

The probability coefficient for a transition of type B is evidently any constant which determines the transition from the particular higher state to a particular lower state. Let this constant be α.

The equilibrium condition can then be written as

$$n_r \beta \frac{N_\nu d\nu}{A_\nu + N_\nu d\nu} = \alpha n_s,$$

which agrees with the fundamental equation (I) if we assume that

$$g_r \beta = g_s \alpha.$$

It is clear that substituting the value of $\frac{n_r}{n_s}$,.one gets Planck's law.

Pauli's case.

This can be easily discussed. The probability of an interaction is as before $\frac{\Sigma r p_r}{\Sigma (r+1) p_r}$. Let $\beta^s_{s'}$ be the probability coefficient for scattering from ν_s to $\nu_{s'}$ if there is an interaction. Let the corresponding coefficients for the inverse process $\frac{\Sigma r p_{r'}}{\Sigma (r+1) p_{r'}}$ and $\beta^{s'}_s$. Then at equilibrium

$$n_r \beta^s_{s'} \frac{N_s d\nu_s}{A_s + N_s d\nu_s} = n_s \beta^{s'}_s \frac{N_{s'} d\nu_{s'}}{A_{s'} + N_{s'} d\nu_{s'}}.$$

Let us assume further that $\beta^s_{s'} = \beta^{s'}_s$; then we obtain the fundamental equation (I).

Einstein and Ehrenfest's case.

The probability of a simultaneous interaction is the product of individual probabilities and therefore equal to

$$\Pi \frac{N_s d\nu_s}{A_s + N_{s'} d\nu_s}.$$

Analogously let $\beta^{s_1, s_2 \dots}_{s_{1'}, s_{2'} \dots}$ be the probability for scattering and let a similar expression hold for the inverse scattering. The rest then easily follows.

Remarks.

Our calculations show that the probability $\frac{N_\nu d\nu}{A_\nu + N_\nu d\nu}$ of an interaction between a particle and a quantum in a radiation field is not simply proportional to the number of the quanta present as one would presume at first sight. It is easy to see that this assumption leads to Wien's law, as realized by Pauli[5]. He found it necessary to assume another form for the probability coefficients. However the form assumed by Pauli and generalized by Einstein and Ehrenfest appears to be completely arbitrary because one cannot easily see how such an expression can be derived. The form suggested here is quite simple and can be justified on the basis of elementary considerations. The necessity of assuming relations between the coefficients themselves is also avoided. It was assumed in deriving the probability coefficients for the interaction (or coupling, as Bohr says) that even in a collision no interaction is as probable as the occurence of any special interaction. This assumption is a fundamental point in the derivation given here. From the classical theory one would expect that there is some interaction whenever a quantum and a material particle come together. Therefore here it is a question of a departure from classical theory. This hypothesis is (as it appears to me), however, very similar to the assumption one usually makes about the stability of stationary states. From classical theory one would expect that no stationary states are possible and that an interaction or coupling (Bohr) between ether and the excited atom and the radiation connected with it is always present. On the other hand, in order to explain spectral emission we must assume it to be possible that no interaction (emission) occurs. Because of the persistence of stationary states we are therefore led to assign to every stationary state a probability coefficient or a mean lifetime.

The assumption made above, that even in a collision no interaction need occur, is quite analogous to the assumption of the stability of stationary states which is so fundamental to Bohr's theory of line emission and can be traced back to the same origin—the probability of existence of the stationary states of the particles. It is interesting to remark in this connection that in the analogous case of a collision of an electron with an atom, experiments show that the electron goes through the atom without changing either the interatomic or its own motion.

The relation $g_r\ \beta = g_s\ \alpha$.

The coefficients or as Bohr calls them, the weights g, are introduced when the valence electron in the atom executes a finite degenerate periodic motion. These numerical coefficients give, according to Bohr, in how many ways from a neighbouring non-degenerate orbit one can arrive at the special degenerate orbit as the limiting case. Let us assume that these coefficients are simply proportional to the number of possibilities of transition from a definite state to the state under consideration ; then the above relation is almost obvious. There are g_s possible ways for a transition from an rth state to the given sth state through absorption. Similarly there are g_r transitions to the lower states as a result of emission. The probability coefficient β has then the factor g_s, the probability coefficient for emission the factor g_r. If we further assume that the probability of a transformation through radiation in a definite way is the same as the probability coefficient for emission when the atom makes transitions to lower states in a definite way, then the relation in question follows immediately[9].

Manindra Physical Laboratory, Dacca University, June 14, 1924.

I consider Bose's hypothesis about the probability of elementary radiation precesses as not appropriate for the following reasons.

For the statistical equilibrium between two states of a Bohr atom the following relation holds, as given by Bose :

$$\frac{n_r}{g_r}\frac{N_\nu}{A_\nu+N_\nu}=\frac{n_s}{g_s}.$$

It follows that the probabilities for the transitions $r \rightarrow s$ and $s \rightarrow r$ respectively on the left and the right hand side of the equation must be proportional to each other. The transition probabilities for a molecule must therefore (if for simplicity we put the statistical weights of both the states equal to 1) behave as $\frac{N_\nu}{A_\nu+N_\nu}:1$. Nothing more can be learnt from the knowledge of the thermodynamic equilibrium. According to my assumptions these probabilities should be proportional to N_ν (i.e. the radiation density) and $A_\nu+N_\nu$ respectively, whereas according to Bose's assumption, $\frac{N_\nu}{A_\nu+N_\nu}$ and 1 respectively.

According to the latter assumption the external radiatian can indeed cause a transition from a state Z_r of lower energy to a state Z_s of higher energy, but not the reverse transition from Z_s to Z_r. This, however, contradicts the generally and rightly accepted fundamental principle that the classical theory should be a limiting case of the quantum theory. According to the latter a radiation field can transfer to a resonator positive as well as negative energy (depending on the phase) and indeed with equal probability. The probabilities of both the transitions must depend on the density of radiation, that is, on N_ν as oppsed to Bose's hypothesis. Planck has discussed in detail in the latest edition of his book on the theory of radiation to what extent the classical theory is the limiting case of the quantum theory.

Secondly, according to Bose's hypethesis a cold body should have an absorbing capacity dependent on the density of radiation (decreasing with it). Bodies in cold state should absorb "non- Wien" radiation to a weaker extent than less intense radiation as defined by the range of validity of Wien's radiation formula. If the behaviour had been such, then it would certainly have already been discovered in infra-red radiation from hot light sources.

A. Einstein

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(rearranged sequentially)

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Messungen der Zersetzungsspannung in nichtwässerigen Lösungsmitteln.

Von

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Aus dem Englischen übertragen.

(Mit 5 Figuren im Text.)

(Eingegangen am 18. 11. 26.)

Über die Zersetzungsspannungen von Salzen in nichtwässerigen Lösungsmitteln liegen nur sehr wenige Untersuchungen vor. Patten und Mott[1]) haben die Kurven der Zersetzungsspannung von Lithiumchlorid in Alkoholen, Aceton und Pyridin unter Berücksichtigung der Vorgänge an Anode und Kathode aufgenommen. Kürzlich haben Müller und Duschek[2]), ferner Müller[3]), Hölzl und andere die Zersetzungsspannung bei Lösungen von Silbernitrat in Pyridin untersucht und die Zersetzungsspannung des Lithiums in Lösungen von Lithiumchlorid und Lithiumnitrat in Pyridin. Diese Untersuchungen ergaben für Pyridin beträchtlich höhere Werte der Zersetzungsspannung als für wässerige Lösungen. Doch waren sie ohne rotierenden Kommutator im Stromkreis angestellt und die elektromotorische Gegenkraft erscheint daher durch das dem grossen Übergangswiderstand der an der Elektrodenoberfläche haftenden Schicht entsprechende Potential vergrössert. Ganz kürzlich hat Newbury in einer Veröffentlichung über „Überspannung und Übergangswiderstände"[4]) festgestellt, dass „die gesamte, dem Durchgang des Stromes von Elektrode zu Elektrolyt sich entgegenstellende Hemmung aus zwei verschiedenen Teilen besteht, wovon der eine reversibel ist (reine Überspannung) und der andere irreversibel (Übergangswiderstände). Die ohne Verwendung eines Kommutators angestellten Messungen der Überspannung pflegen mit einem Fehler von 0·5 bis etwa 2 Volt behaftet zu sein". Pearce und France[5]), sowie

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[1]) Journ. Phys. Chem. 8, 153 (1904); **12**, 49 (1908).

[2]) Monatsh. f. Chemie **43**, 75 (1922).

[3]) Monatsh. f. Chemie **43**, 429 (1923).

[4]) Proc. Roy. Soc. **443** A, 486 (1925).

[5]) Journ. Phys. Chem. **18**, 729 (1914).

Mortimer und Pearce[1]) haben bei Silbernitrat Einzelpotentiale und Gesamtspannung in reinen und gemischten Lösungsmitteln untersucht, und zwar in Methylalkohol, Äthylalkohol, Pyridin und Wasser, sowie in binären Gemischen dieser Stoffe. Dabei fanden sie, dass die Eigenschaften der binären Gemische von Alkoholen mit Pyridin der Mischungsregel folgen, während binäre Gemische von Wasser mit Methylalkohol, Äthylalkohol und Pyridin meist ein ausgesprochenes Maximum oder Minimum der hier in Betracht kommenden Eigenschaften aufweisen. Dies deutet auf eine gegenseitige Beeinflussung der beiden Lösungsmittel hin.

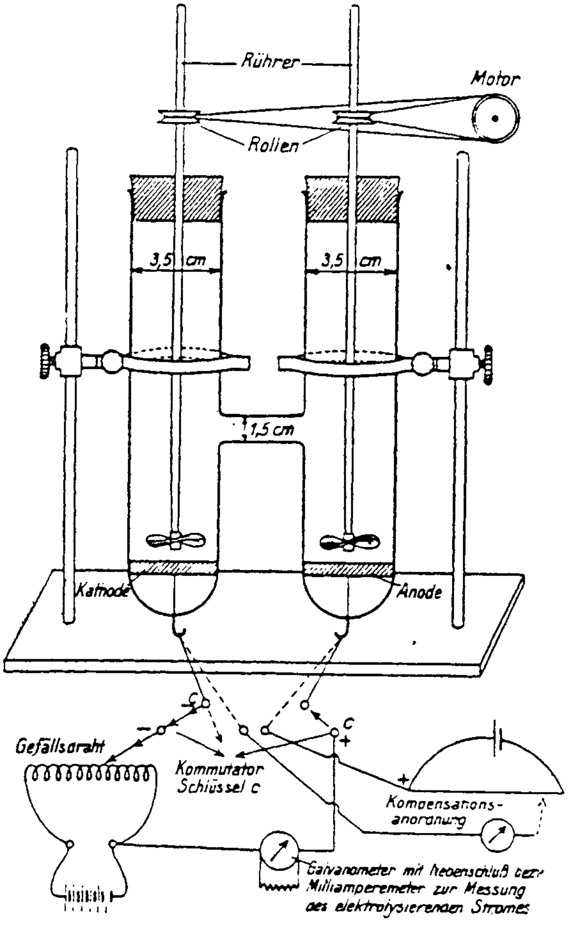

Fig. 1.

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[1]) Journ. Phys. Chem. **21**, 275 (1917).

Im folgenden sollen die Ergebnisse dargelegt werden, die bei der Untersuchung der Entladungsspannung von Chlorwasserstoff und einigen Alkalihalogeniden in reinem Methylalkohol sowohl als auch in Mischungen

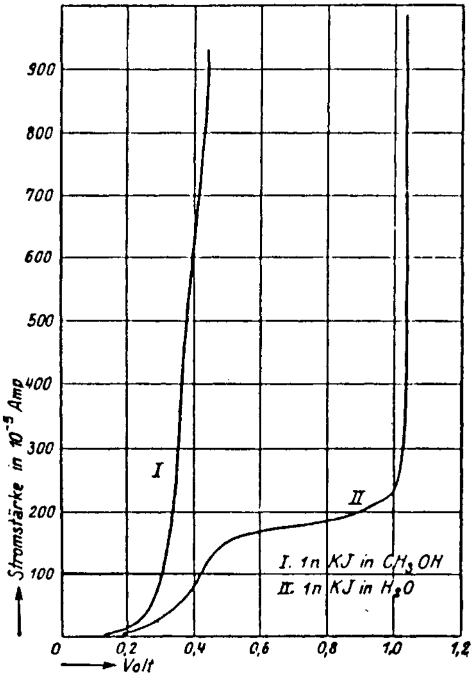

Fig. 2.

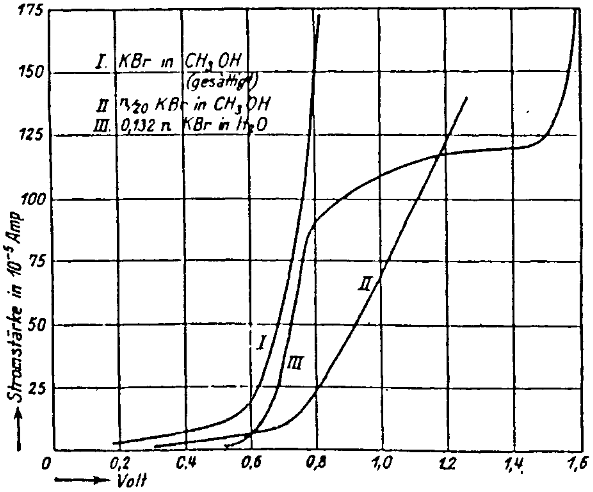

Fig. 3.

von Methylalkohol und Wasser erhalten wurden. Der Methylalkohol war von Merck acetonfrei bezogen, wurde etwa 12 Stunden über Kalk stehen gelassen und dann nochmals destilliert.

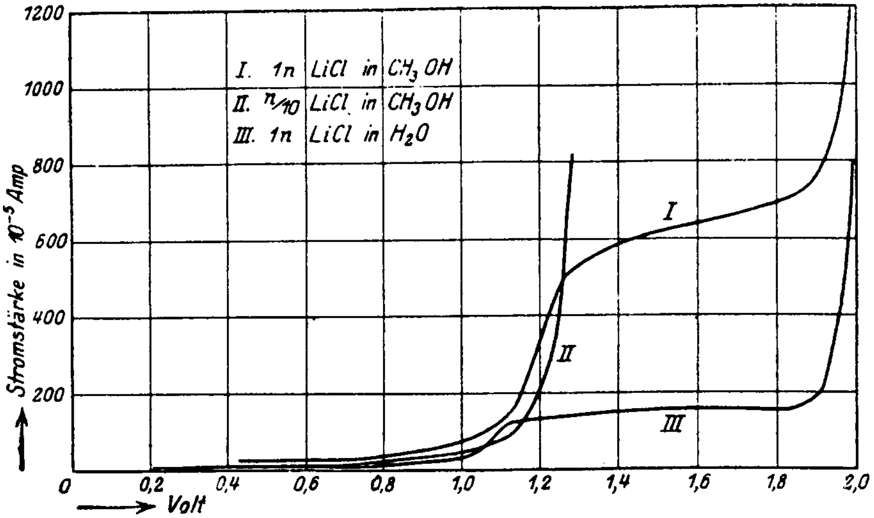

Fig. 4.

Die verwendete Versuchsanordnung erhellt aus Fig. 1. Die Elektroden waren aus dikkem Platinblech und mit Platinschwarz überzogen. Durch den rotierenden Kommutator *C* konnte der Strom fast 3300 mal in der Minute gewendet werden, so dass die Polarisationsspannung der Zelle gemessen werden konnte, nachdem der elektrolysierende Strom bei jeder Wendung nur während 0·009 Sekunden die Zelle durchflossen hatte. Die Stärke des durch die Zelle gesandten Stroms wurde durch ein sorgfältig geeichtes Galvanometer mit Nebenschluss gemessen, für die höheren

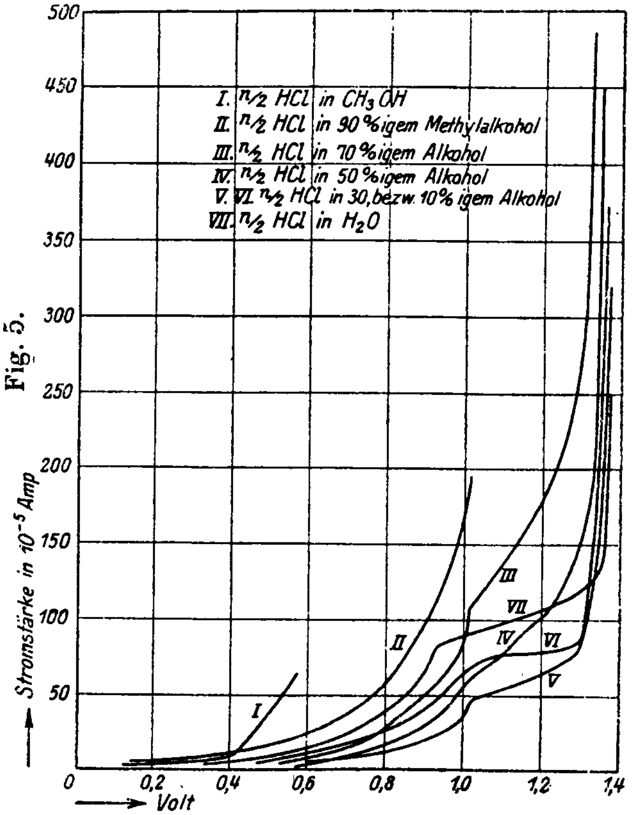

Fig. 5.

Stromstärken wurde ein Präzisions-Milliamperemeter verwendet.

Die Ergebnisse sind aus den Fig. 2, 3, 4 und 5 ersichtlich, worin sie als Stromspannungskurven niedergelegt sind. Die Kurven für die Zersetzungsspannungen in wässeriger Lösung (bei entsprechender Konzentration) sind ebenfalls eigenen Messungen entnommen und stehen in guter Übereinstimmung mit früheren Arbeiten anderer Autoren[1].

Tabelle 1. KJ.

1 norm. KJ in $CH_3 . OH$		1 norm. KJ in H_2O	
Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.	Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.
0·148	5·6	0·194	1·7
0·251	27·8	0·441	110·7
0·294	67·5	0·606	164·7
0·318	92·2	0·794	195·8
0·332	141·8	1·0468	237·0
0·338	181·8	1·116	744·0
0·354	250·0	1·152	1200·0
0·369	372·0		
0·409	610·0		
0·448	970·0		

Tabelle 2. KBr.

0·132 norm. KBr in $CH_3 . OH$		0·066 norm. KBr in $CH_3 . OH$		0·132 norm. KBr in H_2O	
Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp	Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.	Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.
0·194	3·69	0·3	1·12	0·254	0
0·386	7·7	0·493	4·2	0·408	1·4
0·588	15·4	0·726	9·5	0·592	5·32
0·645	32·0	0·800	23·0	0·65	16·8
0·688	58·0	0·836	37·0	0·699	37·8
0·757	90·0	0·933	54·0	0·779	87·8
0·780	104·0	1·019	68·8	0·835	94·5
0·797	135·0	1·074	87·8	0.939	108·0
0·821	162·6	1·173	114·8	1·173	118·0
0·849	206·5	1·24	135·0	1·439	120·0
				1·579	166·0
				1·592	195·8
				1·614	236·3
				1·642	296·0
				1·655	348·0
				1·665	423·0

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[1] Parkin, „Practical methods of Electrochemistry". (Da mir dies Buch nicht mehr zugänglich ist, kann ich die Seitenzahl nicht angeben.)

Tabelle 3. *LiCl.*

1 norm. $LiCl$ in $CH_3 . OH$		0·1 norm. $LiCl$ in $CH_3 . OH$		1 norm. $LiCl$ in H_2O	
Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.	Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.	Entladungsspannung in Volt	Stromstärke in 10^{-5} Amp.
0·444	9·94	0·205	1·12	0·207	0
0·768	34·4	0·586	8·4	0·649	6·72
0·917	54·0	0·794	22·4	0·884	15·26
1·089	104·6	0·925	42·7	1·051	48·5
1·163	155·0	1·024	52·7	1·108	114·8
1·188	297·6	1·113	81·2	1·812	137·8
1·221	372·0	1·138	101·25	1·916	213·0
1·254	469·7	1·16	138·25	1·946	292·0
1·318	565·0	1·177	168·75	1·966	440·0
1·504	613·8	1·18	200·0	1·992	628·0
1·672	660·0	1·19	229·56	2·015	790·0
1·864	730·0	1·212	274·0	2·046	1210·0
1·913	855·6	1·22	311·0		
1·97	967·0	1·229	356·0		
1·984	1116·0	1·236	388·8		
2·0148	1302·0	1·243	443·8		
		1·244	480·0		
		1·254	527·2		
		1·27	625·0		
		1·277	683·0		
		1·287	800·0		

Tabelle 4. *HCl.*

Zersetzungsspannung der Salzsäure in 0·5 norm. Lösungen.
V = Entladungsspannung in Volt, a = Stromstärke in 10^{-5} Amp.

100% $CH_3 . OH$		90% $CH_3 . OH$ 10% H_2O		70% $CH_3 . OH$ 30% H_2O		50% $CH_3 . OH$ 50% H_2O	
V	a	V	a	V	a	V	a
0·123	2·1	0·145	2·8	0·201	1·4	0·173	0·84
0·362	5·6	0·269	4·2	0·538	4·2	0·327	1·54
0·404	9·8	0·404	9·8	0·678	12·6	0·42	2·8
0·436	16·0	0·478	14·0	0·769	20·8	0·703	9·8
0·47	21·0	0·57	22·1	0·829	36·4	0·76	13·7
0·482	27·8	0·619	26·6	0·893	50·6	0·855	23·1
0·499	35·7	0·694	36·4	0·94	62·1	0·956	40·6
0·514	42·0	0·756	48·0	0·947	82·4	1·052	63·4
0·530	49·35	0·807	62·0	1·067	107·4	1·167	94·5
0·543	55·5	0·842	74·0	1·105	143·1	1·243	112·25
		0·893	91·0	1·215	200·5	1·291	141·75
		0·937	119·0	1·26	225·8	1·32	195·8
		0·995	155·0	1·317	353·4	1·341	232·5
		1·037	195·0	1·338	478·9	1·345	274·4
				1·357	618·4	1·356	325·5

Tabelle 4 (Fortsetzung).

30% $CH_3 . OH$ 70% H_2O		10% $CH_3 . OH$ 90% H_2O		100% H_2O	
V	*a*	*V*	*a*	*V*	*a*
0·153	0·84	0·203	0·28	0·34	4·2
0·276	2·66	0·249	1·4	0·604	12·6
0·403	4·2	0·47	5·6	0·929	81·0
0·467	4·2	0·601	8·4	1·116	101·25
0·601	7·7	0·687	16·1	1·298	118·6
0·748	9·8	0·844	33·6	1·355	229·5
0·892	19·8	0·948	54·0	1·376	423·12
1·013	36·4	1·104	77·6	1·4	678·9
1·073	46·2	1·26	81·0	1·404	924·6
1·294	81·0	1·331	148·5	1·434	1353·2
1·332	155·3	1·366	287·7		
1·352	229·5	1·387	451·0		
1·363	372·0	1·404	650·0		
1·386	511·5				
1·393	687·0				

Die Versuchsergebnisse.

Kaliumjodid (vgl. Tabelle 1 und Fig. 2).

Kaliumjodid wurde in ungefähr normaler methylalkoholischer Lösung verwendet. Die gefundene Zersetzungsspannung liegt bei nur 0·32 Volt. Die Anodenflüssigkeit wurde durch die heftige Jodabscheidung schnell angefärbt. In wässeriger Lösung wurde die Zersetzungsspannung von normalem Kaliumjodid bei 1·12 Volt gefunden (Parkin fand 1·14 Volt).

Kaliumbromid (vgl. Tabelle 2 und Fig. 3).

Etwa 0·1 und 0·05 norm. Lösungen von Kaliumbromid in reinem Methylalkohol ergaben eine Zersetzungsspannung von 0·68 bzw. 0·71 Volt. Offensichtlich steigt hier die Zersetzungsspannung mit der Verdünnung. In wässeriger Lösung liegt der Wert für 0·1 norm. Kaliumbromid bei 1·54 Volt (nach Parkin in normaler Lösung bei 1·61 Volt).

Lithiumchlorid (vgl. Tabelle 3 und Fig. 4).

Bei 0·1 norm. Lithiumchlorid zeigt die Kurve nur einen scharfen Knickpunkt und zwar bei etwa 1·2 Volt; bei einfach norm. Lösung, ebenfalls in reinem Methylalkohol, tritt bei etwa 1·90 ein zweiter Knick auf. Nun ergibt eine einfach normale Lösung von Lithiumchlorid im Wasser bei 1·95 Volt einen Knickpunkt. So könnte man vielleicht annehmen, dass in der einfach normalen Lithiumchloridlösung in reinem Methylalkohol Spuren von Wasser enthalten sind, die durch das stark hygroskopische Salz selbst hinein gebracht wurden.

Chlorwasserstoff (vgl. Tabelle 4 und Fig. 5).

Chlorwasserstoff wurde in etwa 0·5 norm. Lösung untersucht und zwar sowohl in reinem Methylalkohol und reinem Wasser, als auch in Mischungen der beiden von verschiedenem Prozentgehalt (10, 30, 50, 70 und 90 % Wasser). In reinem Methylalkohol wurde die Zersetzungsspannung bei 0·41 Volt ermittelt. Bei reinem und 90 % igem Methylalkohol zeigte sich nach Erreichung der Zersetzungsspannung bei 0·4 bzw. 1·0 Volt deutlich die Wirkung des Chlors auf die Anode. Aus Fig. 5 ist die Änderung der Kurvenform bei den Gemischen deutlich zu ersehen.

Erörterung des Einflusses von Gemischen.

Beim Chlorwasserstoff hat der Zusatz von Wasser zu Methylalkohol einen sehr deutlichen Einfluss auf die Zersetzungsspannung dieser Säure. Nun werden durch den Zusatz eines zweiten Lösungsmittels die Eigenschaften der Ionen des Elektrolyten immer dann sehr verändert, wenn zwischen ihnen und dem ersten Lösungsmittel eine engere Bindung möglich war. Dies hat Kraus[1]) an der Leitfähigkeit von Säurelösungen in Alkohol festgestellt, und zwar bei Zusatz von Wasser sowohl als von Salzen, die zur Komplexbildung neigen. Steigt der Prozentgehalt des Wassers in den methylalkoholischen Lösungen der Salze bis zu 30 % und mehr, so tritt nach dem Knickpunkt bei etwa 1·0 Volt noch ein zweiter zwischen 1·3 und 1·33 Volt auf, der der Zersetzungsspannung der Salzsäure in reinem Wasser entspricht. Für 10 % igen Wassergehalt liegt die Zersetzungsspannung bei 1·0 Volt und für reinen Methylalkohol bei 0·41 Volt, wobei die Salzsäure in allen Fällen

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[1]) „Eigenschaften von Systemen mit elektrischer Leitfähigkeit." S. 176 bis 184: Leitfähigkeit von Elektrolyten in Lösungsmittelgemischen. Zeitschr. f. physik. Chemie. CXXV.

0·5 normal war. Demnach ändert sich die Zersetzungsspannung der Salzsäure nicht fortschreitend mit steigendem Wassergehalt, sondern die Mehrzahl der Kurven zeigt zwei ausgeprägte Knickpunkte bei 1·0 und 1·33 Volt. Da in reinem Methylalkohol der Knick bei etwa 0·41 Volt liegt, muss man den bei 1·0 Volt wohl der Bildung eines Komplexes in dem Lösungsmittelgemisch zuschreiben.

Der Einfluss des Wasserzusatzes auf Lithiumchlorid in Pyridin ist von Patten und Mott[1]) untersucht worden. Ihre Resultate sind nicht massgebend, da bei ihren Versuchen die Elektroden sich mit einer isolierenden Schicht überzogen hatten. Die von ihnen erhaltenen Kurven entsprechen in ihrer Gestalt den hier wiedergegebenen. Müller, Hölzl[2]) und andere untersuchten die Entladung des Lithiums in Lösungen von Lithium in Pyridin mit 5 und 10% Wasser. Die erhaltenen Kurven waren vielfach geknickt und unregelmässig. Sie schreiben dies der Abscheidung von Lithiumhydroxyd zu. Lithiumchlorid geht mit Wasser überdies bekanntermassen Molekularverbindungen ein, demnach notwendigerweise auch mit einer Lösung von Methylalkohol und Wasser[3]).

Der Einfluss des Wasserzusatzes auf methylalkoholische Lösungen von Kaliumjodid und Kaliumbromid kann die Gestalt der Kurven nicht sehr ändern, da die Ionen dieser einfach gebauten Salze schwerlich mit dem stark assoziierten Methylalkohol zur Komplexbildung neigen werden. Die Leitfähigkeit wurde jedenfalls durch Wasserzusatz nicht stark verändert.

Erörterung der Stromspannungskurven.

Die Stromspannungskurven in alkoholischer Lösung zeigen sämtlich einen einzigen charakteristischen Knickpunkt, der ziemlich scharf ausgeprägt ist. In wässerigen Lösungen wurden zwei deutlich verschiedene Knickpunkte festgestellt, wovon der eine, und zwar der bei der kleineren Spannung, nicht mit sichtbarer Zersetzung parallel geht, was bei dem zweiten aber untrüglich der Fall ist.

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[1]) A. a. O.

[2]) A. a. O.

[3]) Kraus, a. a. O.

Die Werte der Entladungsspannung wurden in Alkohol sehr viel niedriger gefunden als in Wasser. Andere Untersuchungen an Pyridinlösungen ergaben viel höhere Werte als in wässeriger Lösung. Müller gibt für 1 norm. Lithiumchlorid in Pyridin 3·8 bis 4·0 Volt als Zersetzungsspannung an und für 0·1 norm. Lösung 4·5 Volt, Patten und Mott fanden 4·0 und 4·15 Volt als gesamte Polarisationsspannung der Zelle in Pyridin- bzw. Acetonlösung, während der von uns in Methylalkohol gefundene Wert für 0·1 norm. Lithiumchlorid nur 1·2 Volt beträgt.

Für die hier untersuchten Substanzen zeigt sich in methylalkoholischer Lösung die gleiche Reihenfolge der Zersetzungsspannung wie in wässeriger, sie nehmen in der Reihe $LiCl$, KBr, HCl, KJ ständig ab.

Zusammenfassung.

1. Für Kaliumjodid, Kaliumbromid, Lithiumchlorid und Chlorwasserstoff wurden in reinem Methylalkohol Stromspannungskurven aufgenommen, die Zersetzungsspannungen bestimmt und die Kurven mit denen in wässeriger Lösung verglichen

2. Bei Lithiumchlorid wurde festgestellt, dass schon die Gegenwart von Spuren Wassers die Gestalt der Zersetzungskurve ändert.

3. Es zeigte sich, dass die Werte für die Zersetzungsspannung von Lösungen in reinem Methylalkohol tiefer liegen als in wässerigen Lösungen, aber dieselbe Reihenfolge zeigen, nämlich absteigend von Lithiumchlorid über Kaliumbromid zu Chlorwasserstoff und Kaliumjodid.

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Measurements of the Decomposition Voltage in Non-aqueous Solvents

Susil Chandra Biswas and S. N. Bose

(The text contains 5 diagrams)

(Received on 18.11.26)

Very few findings are available on the decomposition voltage of salts in non-aqueous solvents. Patten and Mott[1] have recorded curves for the decomposition voltage of lithium chloride in various alcohols, acetone and pyridine, taking due consideration of the process taking place at the anode and the cathode. In the recent past Mueller and Duschek[2] and then Muller[3] and Hoelzl and others examined the decomposition voltage in the solutions of silver nitrate in pyridine as well as the decomposition voltage of lithium in solutions of lithium chloride and lithium nitrate in pyridine. These results for pyridine showed considerably higher values for the decomposition voltage than those for the aqueous solutions. They were, however, taken without the installation of a rotating commutator in the circuit; it seems therefore that the opposing electromotive force was increased by the potential corresponding to the large contact resistance of the layer sticking to the surface of the electrode. Very recently Newbury in a publication about the *Overvoltage and the Contact Resistances*[4] has established that 'the total barrier opposing the flow of current from the electrode to the electrolyte consists of two different parts : out of these one (the pure overvoltage) is reversible and the other (the contact resistance) is irreversible. The readings of overvoltage without using a commutator are usually afflicted with an error of 0.5 to 2 volts.' Pearce and France[5] as well as Mortimer and Pearce[6] investigated the individual potentials and the overall voltage for silver nitrate in pure and mixed solvents. These examinations were carried out in methyl alcohol, ethyl alcohol, pyridine and in water as well as in binary mixtures of these substances. These investigations revealed that whereas the binary mixtures of alcohols with pyridine followed the rules for the mixtures, the mixtures of water with

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Fig. 1 : *(see the German original)*

Rührer = stirring rod
Motor = motor
Anode = anode
Kommutator Schlüssel = commutator switch

Rollen = pulleys
Kathode = cathode
Gefällsdraht = step-down coil
Kompensations anordnung = compensator arrangement

Galvanometer mit Nebenschulss bzw. Milliamperemeter zur Messung des elektrolysierenden Stromes = shunted galvanometer milliammeter for measuring the electrolysing current

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methyl alcohol, ethyl alcohol and pyridine mostly showed a pronounced maximum or a minimum in the properties considered here. This fact suggests mutual influence of the two solvents. The results as obtained from the examination of the discharge voltage of hydrogen chloride and a few alkali halides in pure methyl alcohol as well as in mixtures of methyl alcohol and water are presented below.

Fig. 2 *and* **Fig. 3** : *(see the German original)*

(Stromstärke in = Current intensity in)10^{-5} Amp

Fig. 4 *and* **Fig. 5** : *(see the German original)*

(Stromstärke in = Current intensity in)10^{-5} Amp

II. N/2 HCl in 90% methyl alcohol, etc.

Acetone-free methyl alcohol was obtained from Merck. It was allowed to stand for about 12 hours over lime and was then distilled once more.

The experimental set-up used is clear from Fig. 1. The electrodes were made of thick platinum plate and were coated with platinum black. The current could be reversed almost 3300 times per minute with the help of the rotating commutator C. This enabled the measurement of the polarization voltage of the cell when the electrolysing current after every reversal had passed through the cell only for a period of 0.009 seconds. The strength of the current sent through the cell was measured by a carefully calibrated shunted galvanometer. A precision milliammeter was used for measuring stronger currents.

The results are clear from the Figs. 2,3,4 and 5, where they are set down as current-voltage curves. The curves for the decomposition voltages in aqueous solutions (with respective concentrations) were similarly obtained from appropriate measurements and they tally well with earlier works of other authors.[7].

Table 1. ***KI***

1 N *KI* in $CH_3.OH$		1 N *KI* in H_2O	
Discharge potential in volts	Current strength in 10^{-5} Amp	Discharge potential in volts	Current strength in 10^{-5} Amp

(For numerical values, see the German original.)

Table 2. *KBr*

0.132 N KBr in $CH_3.OH$		0.066 N KBr in $CH_3.OH$		0.132 N KBr in H_2O	
Discharge potential in volts	Current strength in 10^{-5} Amp	Discharge potential in volts	Current strength in 10^{-5} Amp	Discharge potential in volts	Current strength in 10^{-5} Amp

(For numerical values, see the German original.)

Table 3. *LiCl*

1 N $LiCl$ in $CH_3.OH$		0.1 N $LiCl$ in $CH_3.OH$		1 N $LiCl$ in H_2O	
Discharge potential in volts	Current strength in 10^{-5} Amp	Discharge potential in volts	Current strength in 10^{-5} Amp	Discharge potential in volts	Current strength in 10^{-5} Amp

(For numerical values, see the German original.)

Table 4. *HCl*

The decomposition voltage of the hydrochloric acid in 0.5 N solutions
V = discharge potential in Volts, a = current strength in 10^{-5} Amp.

100% $CH_3.OH$		90% $CH_3.OH$ 10% H_2O		70% $CH_3.OH$ 30% H_2O		50% $CH_3.OH$ 50% H_2O	
V	a	V	a	V	a	V	a

(For numerical values, see the German original.)

The Results of the Experiments

Potassium iodide (Cf. Table 1 and Fig. 2)

A nearly 1N solution of potassium iodide in methyl alcohol was used. The reading for the decomposition voltage lies in the vicinity of only 0.32 Volt. The anode liquid became quickly tinged by the heavy iodine deposition. The decomposition voltage of 1N potassium iodide in an aqueous solution was found to be 1.12 Volt (Parkin found it to be 1.14 Volt).

Potassium bromide (Cf. Table 2 and Fig. 3)

Approximately 0.1 and 0.5 N solutions of potassium bromide in pure methyl alcohol yielded decomposition voltages of 0.68 and 0.71 Volt respectively. Apparently the decomposition voltage increases here with dilution. In the aqueous solution it was in the vicinity of 1.54 Volt for 0.1N potassium bromide (according to Parkin it is 1.61 Volt for a 1N solution).

Lithium chloride (Cf. Table 3 and Fig. 4)
In the case of a 0.1N lithium chloride solution the curve shows a sharp bend in the neighbourhood of about 1.2 Volt. In the case of a 1N solution in pure methyl alcohol, a second bend similarly appears in the neighbourhood of 1.90 Volt. A 1N solution of lithium chloride in water shows a bend in the vicinity of 1.95 Volt. One could therefore assume that there are traces of water present (coming from the strongly hygroscopic salt) in a 1N lithium chloride solution in pure methyl alcohol. On the contrary, the majority of the curves show two marked bends in the vicinity of 1.0 Volt and 1.33 Volt. Since in the case of methyl alcohol the bend lies in the neighbourhood of 0.41 Volt, one must attribute the bend in the vicinity of 1.0 Volt to the formation of a complex in the mixture of solvents.

The influence of adding water to lithium chloride in pyridine was examined by Patten and Mott[9]. Their results are not decisive as the electrodes became coated in their experiments with an insulating layer. The shapes of the graphs produced by them correspond with the ones reproduced here. Mueller, Hoelzl[10] and others examined the discharge of lithium in solutions of lithium in pyridine with 5 and 10% water content. The graphs thus produced contained several bends and were irregular. They attribute this fact to the deposition of lithium hydroxide. Moreover, it is well known that lithium chloride enters to form a molecular compound with water and therefore necessarily also with a solution of methyl alcohol and water[11].

The addition of water to the solution of potassium iodide and potassium bromide in methyl alcohol cannot influence the curve in such a way as to change its shape very much, because the ions of these simply built salts will be scarcely inclined to form complexes with the strongly associated methyl alcohol. In any case the conductivity was not much changed because of the addition of water.

Discussion of the Current Voltage Curves

The current voltage curves in alcoholic solutions show a single sharply pronounced characteristic bend. In aqueous solutions two distinctly different bends were established of which the one with the lower voltage does not go along with a visible decomposition which is unmistakably the case with the other one.

The values for the discharge potential were found to be very much lower in alcohol than those in water. Other studies on pyridine solutions have yielded much higher values than the ones for an aqueous solution. Mueller states 3.8 to 4.0 Volt to be the decomposition voltage for a 1 N lithium chloride solution in pyridine. For a 0.1 N solution he states it to be 4.5 Volt. Patten and Mott found 4.0 Volt and 4.15 Volt to be the total polarization potential of the cell in pyridine and acetone solutions respectively. In contrast, the value for 0.1 N lithium chloride in methyl alcohol was found by us to be only 1.2 Volt.

The decomposition voltage for substances studied here show the same sequence in a methyl alcohol solution as in an aqueous solution. They decrease steadily in the sequence of LiCl, KBr, HCl, KI.

Summary

1. Current voltage curves were obtained for potassium iodide, potassium bromide, lithium chloride and hydrogen chloride in pure methyl alcohol. The decomposition voltage was determined ; the curves were compared with those for aqueous solutions.
2. It was established for lithium chloride that even the presence of traces of water changes the shape of the decomposition voltage curve.
3. It was seen that the values for the decomposition voltage in solutions of pure methyl alcohol are lower than those in aqueous solutions but show the same sequence, namely descending from lithium chloride to potassium bromide, hydrogen chloride and potassium iodide.

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(rearranged sequentially)

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Beryllium Spectrum in the Region λ 3367-1964. *By* S.N. BOSE, *Professor of Physics, Dacca University, and* S.K. MUKHERJEE, *Assistant Lecturer in Physics, Dacca University**.

[Plates II. & III.]

THE beryllium spectrum is remarkable owing to the small number of lines that have been observed in the visible and the ultra-violet. The earlier measurements are somewhat conflicting. Exner and Haschek could not find some of the lines observed by Rowland and Tatnall †. Glaser ‡ investigated the spectrum by sparking between the metal tips, but he also could not corroborate the earlier observations.

Recently Millikan and Bowen § subjected the spectrum to a thorough analysis by their method of hot spark, and extended it up to 1943 Å.U. They classified some of the important lines, and gave the term values both of Be II and Be I.

It was with a view to study the spectrum thoroughly, under different conditions of excitation, that the present work was undertaken.

The instrument that we have used almost exclusively for the work is Hilger's Quartz Spectrograph, type E 1. In the near ultra-violet between the regions 2400-1850 it is almost an ideal instrument for spectrum analysis, combining a high dispersion with a high light-value. We have extended our observations to the visible region; but here, because of the low dispersion, the measurements were not more accurate than by ·1 Å.U., though we could very easily identify the lines and thus verify the observations of the previous workers.

Most of the observations were made on the arc spectrum, and obtained by feeding the nitrate or the metal in a carbon arc. We have used both ordinary and Hilger's extra pure carbon rods, and the presence of traces of iron has been an advantage in furnishing suitable standard lines in the different regions. In the extreme ultra-violet we have used copper arc and spark as comparison spectra, and in some cases the silicon lines almost always present in carbon spectra were helpful in making accurate measurements.

The plates used were Ilford Empress and Ordinary up to 2175. In the extreme ultra-violet we tried sensitization with machine oil, but the lines were a little broad. As our aim was to resolve very narrow doublets, we could only get the best results by using Schumann plates as supplied by Hilger.

The materials used were, at first, certain old samples of beryllium nitrate and metal from Merck's, which were found to contain lead, aluminium, and silver as impurities. Later, owing to the kindness of Professor Mark, of Badische Anilin Fabrik, we were enabled to get a different sample of the metal. But this was found to contain traces of rare earths, notably scandium and yttrium. By a comparison of the different samples the lines due to contamination were eliminated, and we give our results tabulated below (*vide* Table I.).

As may be seen, we could go almost to the same limit as Millikan and Bowen by their hot-spark method, and, incidentally, we have discovered that the following lines, 2351, 2175,2126, 2056, and 2033, given as singlets by Millikan and Bowen, are clearly doublets of approximate wave-number difference 2.6. Most probably these are triplets due to the triplet P-terms, and our spectrograph could only separate P_1 from $P_2 + P_3$. The wave-number difference thus measured is the distance of P_1 from the centre of gravity of P_2- and P_3- lines, and as such has a slightly higher value.The lines 1998 and 1964 just appeared as doublets, but owing to their hazy character the measurements of the doublet separation were not possible. We could verify the earlier measurements of Rowland and Tatnall as well as the line of Glaser, viz. λ 4672·9. Incidentally, we have discovered a few new lines whose wave-lengths are given in Table I. The line 3019 appeared as a doublet, the line 2986 as a triplet, and the lines 3110 and 2738 as singlets.

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* Communicated by the Authors.

† Kayser, *Handbuch d. Spectro.* vol. v.

‡ Glaser, *Ann. d. Phys.* (4) lxvii. pp. 73-88 (1922). § Millikan and Bowen, Phys. Rev., Aug. 1926.

We have made some observations by feeding a fair amount of metal or the salt in the arc, and we could observe that the lines 3321, 2651, 2494, and 2175 with their components were reversed in some cases, thus showing that the *p*-level is probably the common origin of the lines as classified by Millikan and Bowen. In rare occasions, however, the line 3131, usually attributed to Be II, could be reversed. The line 2348 (1S-2P) is always reversed in the arc.

We have tried the spark spectra in air, where, owing to rapid oxidation, we could make very little progress. We have also tried sparking between the tips of the metal, enclosed in a partially evacuated vessel with quartz window; and even then we could not go beyond 2175. In this way,however, we have got an interesting band spectrum, which we believe to be due to beryllium and which consists of bands degraded to the red. The fine structures of the band have not yet been measured, and we give in Table II. the measurements of the edges.

There does not seem to be any line of Be between 4700 up to the extreme red. We are now continuing the work in the infra-red, and hope to be able to corroborate the solitary work of Theo Volk * in the region.

The following table gives the wave-lengths (in I.a, vac.) of the lines between λ3367-1964, with the frequency difference between the components, and also the lines (in I.a) of Rowland and Tatnalll, together with the few lines marked * observed by us as new :—

TABLE I.

Bose & Mukherjee	Rowland & Tatnall I. *a*.	Bowen & Millikan I. *a*, vac.	$\delta\nu$
3367.55	3367.579	-	-
*3110.91	-	-	-
*3019.54	-	-	-
*3019.33	-	-	-
2986.63	-	-	-
2986.44	2986.426	-	-
2986.08	2986.057	-	-
2898.26	2898.242	-	-
*2738.08	-	-	-
2351.40 2351.54	-	2351.50	2.50
2175.72 2175.84	-	2175.72	2.64
2126.30 2126.42	-	2126.57	2.66
2056.65 2056.77	-	2056.71	2.60
2033.23 2033.34	-	2033.43	2.66
1998(a)	-	1998.19	
(a)	-	1964.81	

The lines marked (a) appear in plates, but their measurements are unsatisfactory owing to the absence of standards in this region.

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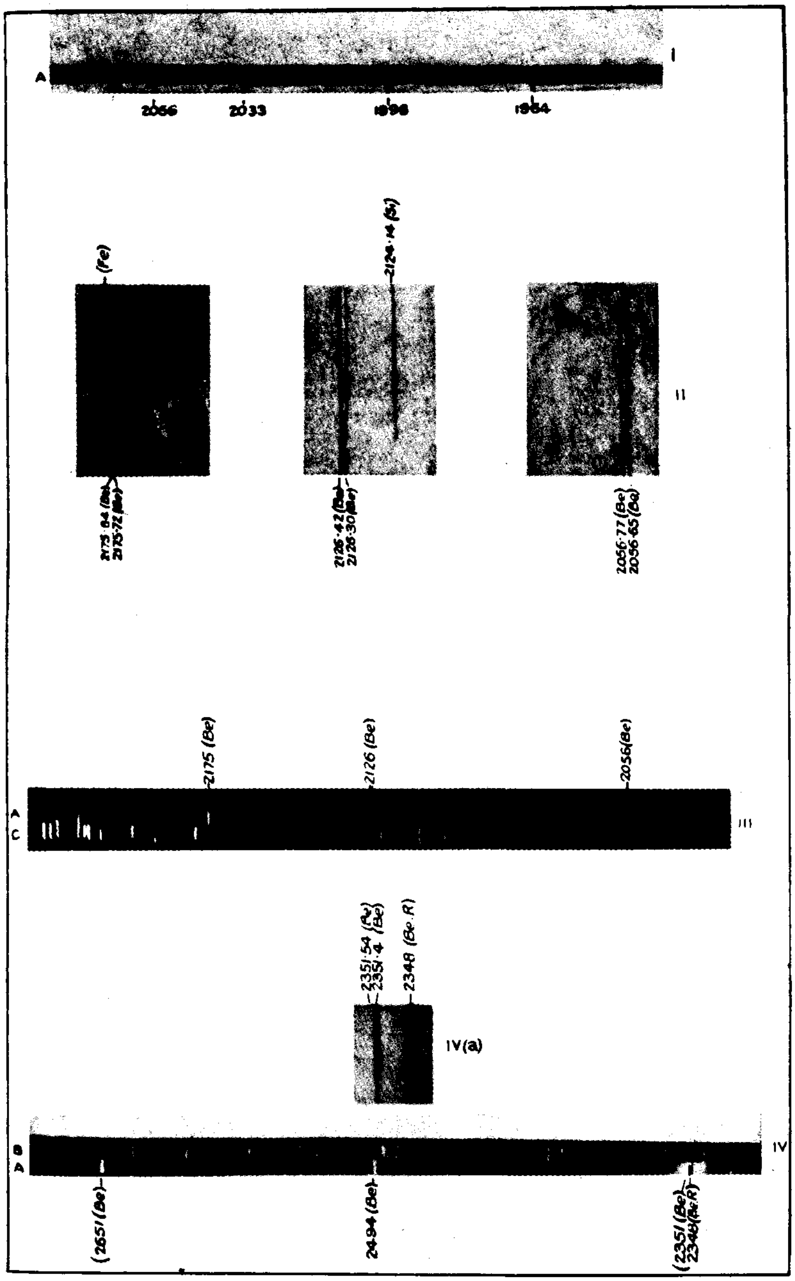
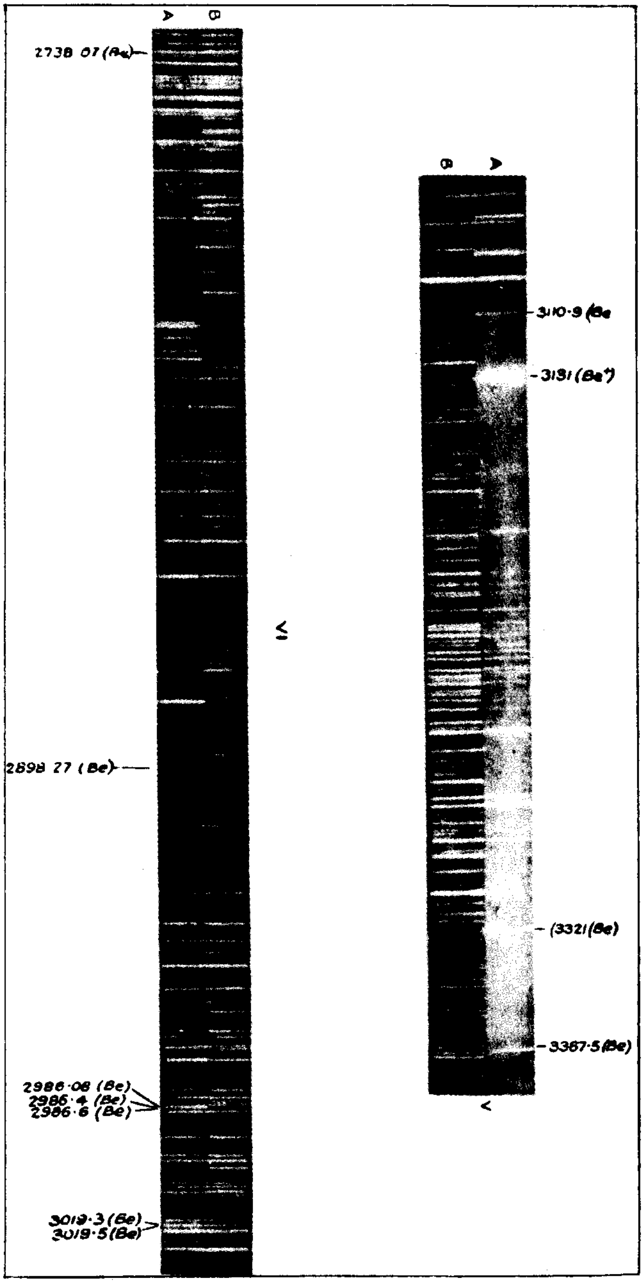
*Dissert. Tübingen, 1924 (Dresden bei Teubner). 17 pp.

Bose & Mukherjee. **Phil. Mag. Ser. 7, Vol. 7, Pl. II.**



A. Carbon arc fed with Be Salt.
B. Iron arc.

Bose & Mukherjee. Phil. Mag. Ser. 7, Vol. 7. Pl. III.



A. Carbon are fed with Be Salts.
B. Iron Are.

In the accompanying plates (Pls.II. and III.) I. and III. are contact-prints from Schumann plates; II. and IV. (a) are microphotographic enlargements for showing the fine structure; IV., contact-print from ordinary plate; V. and VI. are slightly enlarged prints to show the new lines.

The following table gives the wave-lengths (in I.a) of the edges of the band; those marked (a) are fully developed with their fine structures:—

TABLE II.

1..........	2474.2	6..........	2285.8
2 (a)......	2446.0	7..........	2260.8
3 (a)	2419.2	8..........	2238.3
4 (a)......	2325.1	9..........	2221.7
5 (a)	2299.4	10(a).....	2189.0

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Section of Mathematics and Physics.

President :—Prof. S. N. Bose, M.Sc

Presidential Address.

TENDENCIES IN THE MODERN THEORETICAL PHYSICS.

The ultimate aim of Scientific inquiry is to arrive at a minimum number of hypotheses which will explain the maximum number of facts. The hypotheses should obviously not contradict one another. At the present moment, however, we see two contradictory theories, in the domain of Physics. On the one hand we have the classical theory based on the dynamical laws of Newton. On the other hand we have the Quantum theory first introduced by Planck, which has been differently formulated by different Scientists; while the Classical Theory explain satisfactorily all problems relating to motion and interaction of big masses as well as the problem of propagation of radiation, the Quantum Theory has succeeded with the help of a few principles in co-ordinating a large amount of experimental material accumulated in the various domains of spectroscopy, X-rays, etc. It has succeeded best in all problems dealing with the ultimate constitution of matter, or in problems dealing with the interaction of matter with radiant energy. As a result of work of the last twenty-five years, we seem to be much nearer to understanding the problem of matter. The periodic classification of Mendeljeff does not now appear as an unexplained riddle and the huge amount of spectroscopic material can now be classified and explained with the help of a few fairly simple principles. The task that faces the Physicists to-day is how best to harmonise the seeming discord of the two theories, which are at the present moment utilised to explain the physical phenomena. It is a difficult task, and the first step towards fulfilment will be to place in a clear light the differences that characterise them: for this purpose it is necessary to trace the history of the development of our physical ideas indicating the various points where fresh hypotheses had to be introduced before the next move towards progress could be made.

Theoretical Physics may be said to have begun its career as a Science with the formulation of the famous laws of motion by Newton. The mathematical formulation of this principle leads to a series of differential equations in which we equate certain quantities which depend upon the state and nature of the body under investigation with certain other quantities which we interpret as forces arising out of the interaction of other bodies. The solution of these differential equations introduces a certain number of arbitrary constants which depend upon the initial state of the system investigated. These constants once known determine completely the subsequent history of the system under observation. Newton's original equations involved the use of the cartesian co-ordinate system, but very soon these equations were transformed into a form in which the arbitrary character of the co-ordinates was removed, and finally Hamilton and Jacobi introduced the characteristic function, which reduced the problem of solving the equations of dynamics to the solution of a single partial differential equation. Simultaneously, with the help of the concept of the Variational Calculus, we arrived at the celebrated Hamilton's Principle, by which the whole problem of dynamics was reduced to the problem of making a certain integral an extremum, within certain suitable limits.

The writing down of the set of differential equations, or of the single partial differential equation of Jacobi requires however the knowledge of the laws of interaction of various bodies on one another. The exact formulation of these laws seem therefore to be the immediate aim of the earlier classical Physicists. These laws once known allow us to apply to any problem the general dynamical methods, whereby the problems of Physics are reduced to problems of pure mathematics. In the laws of gravitation, in the law of Coloumb and of Ampere, we have some classical examples ; in every case it is endeavoured to express the force in terms of the relative co-ordinates, and the velocities of the interacting masses.

The laws of dynamics were originally formulated to explain the motion of observable bodies. When with the progress of knowledge the discrete nature of the constitution of matter was evident, the natural endeavour has then been to extend to those ultimate particles (the atoms, protons, electrons), the same dynamical laws which have been so successfully applied to the study of big masses. As the aim of Physics is to explain the observed physical phenomena, in terms of the motion of the ultimate particles, a consistent application of the dynamical methods has been responsible for the whole structure of classical Physics.

Before the methods of classical dynamics could be applied important developments in two directions had to be made. The problem of specifying the law of interaction of the different particles, when we have to take account of the immense number of the constituent particles raised difficulties which were resolved and the problems made more amenable to analysis by the introduction of the conception of the Field. It was discovered that instead of basing the description of interaction on the various laws, (tormulated on the action-at-a-distance basis), a much simpler and a better treatment could be obtained by the introduction of certain auxiliary magnitudes—The scalar or the vector potential, or The Electric or Magnetic intensities. These quantities vary with the position in space, as well with time. The characteristic functions determining the field were found in the case of electro-magnetism to satisfy a certain linear set of partial differential equations, whereas the relation between the magnitudes introduced, and the reacting masses may be expressed by a certain type of equation of the second order. These auxiliary magnitudes thus came to be looked upon as having their seat in a hypothetical medium which was postulated to pervade all space, and the partial differential equations which connect them came to be looked upon as related in some way to the physical properties of the hypothetical medium, the Ether. By the introduction of the concept of energy and momentum density, the abstract conception of ether seemed to gain in substantial reality. This was further strengthened when it was shown that it was possible to bring the partial differential equations in line with the usual dynamical equations, in as much as they appeared to be deducible from the Hamilton's principle if certain quantities were identified with the potential and the kinetic energies of the medium. Thus we come to the classical conception of Ether, and the electro-magnetic equations of Maxwell which served as the starting point of the modern Electron-Physics. Gravitation, however, did not immediately fall in line with the other field theory, until the formulation of the generalised Relativity-Principle by Einstein. Of this we shall have occasion to speak later on.

There was however another difficulty which stood in the way of immediate application of the classical methods to the problems of Physics. The dynamical laws seem at first applicable to the motion of the ultimate particles, which themselves however always escape direct observation. What we observe is to be looked upon as the resultant of a large number of elementary events; and the measurable microscopically sensible magnitudes cannot also be regarded as determining uniquely the component microscopic elements. Thus, from the classical standpoint, the necessary magnitudes for uniquely characterising the state of the system remain largely undetermined, and the Physical laws connecting observable things cannot be regarded as immediately deducible from the dynamical laws, at any rate without the introduction of further hypotheses. The method of statistical mechanics was developed to tackle this fundamental difficulty. The earlier method consisted in regarding the ultimate particles as practically independent systems. Each system is characterised by certain values of co-ordinates and momenta. Under their mutual interaction, Space and momenta cordinates of the individual systems are supposed to vary within certain limits. The actual physically observable magnitudes are determined by the distribution of the systems among the different physically possible states. Mathematically this distribution is visualised by the distribution of the representative points of the individual systems in a phase space, which is supposed to be decomposed into a certain number of elementary phase-cells. The actual position of any particular representative point inside the phase-cell, is supposed to have no effect on the value of the phenomenologically observable magnitudes. The equilibrium-state of a physical system can thus be related to a particular distribution of the representative points in the phase cells which occurs most frequently, or remain longest during the period of observation. By this way the important physical idea of entropy can be connected wtih the probability of distribution in the Phase-Space, and important Thermodynamical laws can be looked upon as certain statistical laws which are necessary consequences of the very large number of individual components. When the results of classical dynamics are combined with this concept of statistics, we get certain very general laws, regarding the partition of energy. Though at first, these results seemed to agree with the facts observed, subsequent and more exact experiments have shown them to be erroneous.

I have already referred to the equations of classical mechanics which determine the motion of a material particle, as well as the field equations, which connect the vector magnitudes serving to describe the electro-dynamical field. Though it was attempted to bring these equations in line with the dynamical laws by showing that they also can be deduced from Hamilton's Principle, a fundamental difference between the two sets of equations was clearly brought out during the early part of this century.. The dynamical equations of Newton possess an invariant group of transformation, the Galilean Group, which expresses the equivalence of all inertial systems as frames of reference for the description of motion. The field equations of Maxwell however, have a different invariant group, the Lorentzian group. The space and time co-ordinates enter symmetrically into these equations, or rather the space-time symmetry can be brought out by introducing after Minkowski an imaginary time ict., as the fourth co-ordinate. The Lorentzian group of transformation can be represented by a rotation of axes in a four dimensional orthogonal space, which leaves the invariant distance unaltered. By extending the conception of vector quantities of the usual three dimensional space to four dimensions, the field equations can be given an invariant form. The recognition of this formal equivalence of the time and space co-ordinates ushers in the ideas of relativity. This equivalence now-a-days is regarded as axiomatic, and furnishes a test which all physical laws have to satisfy in order to be exact. By generalising this conception further, and by giving up the condition of orthogonality Einstein was able to present a field theory of gravitation. The two different field theories have however remained essentially distinct though attempts have repeatedly been made to fuse them into one single unitary theory.

I have now indicated to you the principal steps by which the classical theory arrived from its first elementary stage to the present developed state. I shall now tell you of the difficulties that arose successively and which led to the formulation of the quantum theory. The conceptions of energy and momentum have been early deduced from the dynamical laws and have played very important roles in the subsequent development of the Science. For example the Hamilton-Jacobi differential equation requires for the mathematical expression the writing down of total energy of the system in terms of the momenta, whereas the integrand in the Hamilton's Principle is the difference between the potential and the kinetic energy of the system. The introduction of the concept of energy seems necessary for the applicability of the general dynamical laws, and we have seen how the field equations can be made compatible with the dynamical laws by interpreting a certain magnitude as the energy-density of Ether. As a necessary corollary to the equations however it follows that there will be a continuous interchange of energy between the ultimate particles composing matter and the surrounding radiation field. The principle of equipartition of energy which follows as a necessary consequence of the idea, leads us however to entirely wrong results when applied to the problem of equilibrium of radiation-field with matter. In order to explain the distribution of energy in the black-body radiation spectrum, as well as to explain the problem of generation of radiation we have been compelled to make certain assumptions regarding the constitution of the radiation-field, and about the equilibrium states of material particles, which directly contradict our classical ideas. This has led to the introduction of the Quantum theory in Physics. The energy in the monochromatic radiation field, has to be assumed to exist in definite quanta. The study of the constitution of matter has led us to the conception of the atom as a comparatively stable structure built out of the nucleus and electrons. These electrons according to the earlier formulation of the quantain theory are to be supposed as rotating in certain characteristic orbits ; whereas such a constitution on the classical theory will necessarily continually radiate and lose energy, we have to assume here that such a structure keeps generally intact its energy-content and also that there are a series of discrete radiation-free states characterised by a discrete series of values of the energy content. An interchange of energy can only take place, when the atomic system passes from one distinct state to another resulting in an emission or an absorption of radiation. The passage of the atom from a higher energy state to a lower is associated with an emission of a monochromatic radiation whose frequency is given by the relation

$$W,-W'=hn.$$

where W, and W' are the characteristic energies of the two states.

Modern development of spectroscopy seems to indicate that the state of a system may be defined by the introduction of certain sets of quanta members, each set being associated with a definite constituent electron of the atomic system. No two electrons of the atomic system may have the same group of the quanta-numbers, and the successive building up of the different atoms may be looked upon as a gradual increase in the number of electrons in the outer *orbit*, each electron coming to occupy a place characterised by a different group of quanta numbers.

These few principles which are so contradictory to the ideas of classical dynamics have proved to be invaluable in the development of modern Physics. To them we owe the scientific classification of the spectroscopic terms, explanation of the Periodic classification, etc., etc. The direct and simple way in which the different problems, which have long baffled the attempt of classical physics, find their explanations in the new theory shows that it contains a large element of truth, and the most important problem of the present time is to find a way of reconciling the two theories. It has become evident also that no one theory will be able to entirely replace the other. For example the conception of the field, which has been developed originally from the classical standpoint, seems to contain also a great element of truth. The large mass of experimental material connected with the problem of propagation of radiation, seems to find in the field theory a very simple and direct explanation, which appear so difficult to explain from the Quantum-theoretical standpoint.

The solution of the dilemma seems to lie in suitably uniting the field and the quanta-theories, which will ultimately form a more general theory of which the two theories may then be regarded as limiting cases. We have in recent years already some indications of tentative attempts in the same direction. One may attempt to visualise the radiation-field as simply determining the interaction of the different particles and give up the classical concept of energy and momentum-density in Ether. Consistent with our ideas of the quantum theory we may regard that the energy in the field is only discretely distributed throughout the space, (somewhat after the ideas of J. J. Thomson). The motion of these quantas of energy may be supposed to be guided as it were by the field, and it may be supposed to take place along the direction of the Poynting

Vector. The same field which guides the motion of the quanta may be also supposed to control the motion of the electrons in the different orbits. The nature of the field itself should be determined by some equations depending upon the distribution of the charges. Such a field theory will have as its task the explanation not only of the fundamental relation E=hv, but also of the occurrence of the discrete quanta-states and the different quanta-members.

That the idea of discrete energy-states is not inconsistent with a continuous field-theory seems to follow from the recent developments of the wave-mechanics (Schrodinger, Dirac). In the classical theory the impossibility of the existence of stationary electron-orbits follows from the result that an accelerated electron gives rise according to it to an electro-magnetic field such that the surface integral of the Poynting Vector at a great distance does not vanish. The result is interpreted as indicating a continuous loss of energy from the moving electron which makes stationary orbits impossible. One can think however, that if there are other electrons present the resultant field may be such that the electron may in the average gain as much energy from the surrounding field as it loses by radiation, or in other words the resultant electro-magnetic field due to all the moving electrons may be such that the integral of the Poynting-Vector vanishes. Such a hypothetical solution of the field-equations which make the field vanish at infinity may be thus likened to a stationary vibration rather than to a divergent wave-train. It is clear, however, that the distribution of the charges will have to be suitably made and their motions definitely related in phase with one another in order that such a thing may be possible. The electro-magnetic field may then be supposed to be in resonance after de Broglie. What Schrodinger succeeds in establishing may be likened to finding exactly such a law of distribution of charges. The quanta conditions then may be fittingly compared with the resonance conditions. Dirac seems to have succeeded in deducing a set of four linear partial differential equations which correspond to facts more closely. The four functions or rather eight of them give us the distribution of the hypothetical charges and currents in the electro-magnetic field necessary to establish resonance-conditions. The field itself in these two theories determines also the distribution or rather the statistically equivalent distribution (in space as well as in velocity) of the charges in the field.

One may hope that the final solution of problem may be found and the proper set of equations which will determine the radiation-less field may be determined with the help of the ideas of the generalised relativity theory. If an unitary field theory which could explain gravitation and electro-magnetism could be found, one would hope to obtain thereby not only the solution of the quanta-problem, but perhaps also of a more fundamental problem, the relation of charge with gravitational mass, and explanation of the difference in mass of the proton and the electron.

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ON THE COMPLETE MOMENT-COEFFICIENTS OF THE D^2-STATISTIC.

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The D^2-statistic was defined by Mahalanobis[1] to measure the divergence between two statistical populations, and the moment-coefficients were also calculated by him by approximative methods. Rajchandra Bose[2] has found the exact distribution of the D^2-statistic* which can be expressed in terms of Bessel functions. He has used the actual distribution function to obtain the moment-coefficients, and has shown that the results previously obtained by Mahalanobis are exact. In the present note a recurrence-formula for the D^2-statistic has been obtained directly without assuming the distribution, and certain properties of these moment-coefficients have been investigated.

2. Let $(a_1, a_2, \ldots a_p)$ and $(a'_1, a'_2, \ldots a'_p)$ be the observed mean values in two samples Σ_1 and Σ_2 of size n_1 and n_2 respectively drawn from two normal p-variate correlated populations with mean values $(\alpha_1, \alpha_2, \ldots \alpha_p)$ and $(\alpha'_1, \alpha'_2, \ldots \alpha'_p)$ respectively. The distribution of the differences in observed mean values $(a_1-a'_1, a_2-a'_2, \ldots a_p-a'_p)$ can be written in the form :—

$$\text{Const.} \times e^{-\frac{n}{p\beta}\left[\beta_{11}\{(a_1-a'_1)-(\alpha_1-\alpha'_1)\}^2+\ldots 2\beta_{12}\{(a_1-a'_1)-(\alpha_1-\alpha'_1)\}\{(a_2-a'_2)-(\alpha_2-\alpha'_2)\}+\ldots\right]}$$

$$\times d(a_1-a'_1),\, d(a_2-a'_2)\ldots d(a_p-a'_p) \qquad \ldots \quad (2.1)$$

where β and β_{ij} $(i, j = 1, 2, \ldots p)$ are certain functions of the population variances and co-variances whose values have been given by Rajchandra Bose[3] and

$$\frac{2}{n} = \frac{1}{n_1} + \frac{1}{n_2} \qquad \ldots \quad (2.2)$$

Then the D^2-statistic is defined by

$$D^2 = D_1^2 - \frac{2}{n} \qquad \ldots \quad (2.31)$$

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* In this paper p and n have been used in the place of P and $\bar{n}$ in the earlier papers of P. C. Mahalanobis and Raj Chandra Bose. The D^2-statistic was intended to be and was defined as a quantity determined entirely by the sample values of the variates. Raj Chandra Bose has investigated the exact distribution and S. N. Bose the moment-coefficients of a modified form of the D^2-statistic in which the population values of the variances and co-variances have been substituted for the corresponding sample estimates.—*Editor, Sankhyā.*

and
$$D_1^2 = \frac{1}{p\beta} \mathop{\mathrm{S}}_{i,j=1}^{i,j=p} [\beta_{ij}(a_i-a_i')(a_j-a_j')] \quad \ldots \quad (2.32)$$

The population value of D^2 is Δ^2 which is defined by

$$\Delta^2 = \frac{1}{p\beta} \mathop{\mathrm{S}}_{i,j=1}^{i,j=p} [\beta_{ij}(\alpha_i-\alpha_i')(\alpha_j-\alpha_j')] \quad \ldots \quad (2.4)$$

3. If we put

$$(a_i-a_i') = x_i, \text{ and } (\alpha_i-\alpha_i') = \xi_i$$

we can write

$$D_1^2 = \frac{1}{p\beta} [\beta_{11}x_1^2+\beta_{22}x_2^2+ \ldots 2\beta_{12}x_1x_2+ \ldots] \quad \ldots \quad (3.1)$$

$$\Delta^2 = \frac{1}{p\beta} [\beta_{11}\xi_1^2+\beta_{22}\xi_2^2+ \ldots 2\beta_{12}\xi_1\xi_2+ \ldots] \quad \ldots \quad (3.2)$$

If we now use a linear transformation which changes the quadratic form

$$\frac{1}{2\beta} \mathop{\mathrm{S}}_{i,j=1}^{i,j=p} (\beta_{ij}x_ix_j) \quad \text{to} \quad (y_1^2+y_2^2+ \ldots y_p^2) \quad \ldots \quad (3.3)$$

then the same transformation will change

$$\frac{1}{2\beta} \mathop{\mathrm{S}}_{i,j=1}^{i,j=p} (\beta_{ij}\xi_i\xi_j) \quad \text{to} \quad (\eta_1^2+\eta_2^2+ \ldots \eta_p^2) \quad \ldots \quad (3.4)$$

where $(\eta_1, \eta_2, \ldots \eta_p)$ are connected with $(\xi_1, \xi_2, \ldots \xi_p)$ by the same linear transformation which connects $(y_1, y_2, \ldots y_p)$ and $(x_1, x_2, \ldots x_p)$

Also $\frac{1}{2} \mathop{\mathrm{S}}_{i,j=1}^{i,j=p} (\beta_{ij}x_i\xi_j)$ will be transformed to $(y_1\eta_1+y_2\eta_2+\ldots y_p\eta_p)$ … (3.5)

We can then write

$$\tfrac{1}{2}p\cdot D_1^2 = y_1^2+y_2^2+ \ldots y_p^2 \quad \ldots \quad (3.6)$$

$$\tfrac{1}{2}p\cdot \Delta^2 = \eta_1^2+\eta_2^2+ \ldots \eta_p^2 \quad \ldots \quad (3.7)$$

Equation (2.1) can then be written in the form :—

$$\text{Const.} \times e^{-\frac{np}{4}[D_1^2+\Delta^2-2D_1.\,\Delta\cos\theta]} \times dV \quad \ldots \quad (3.8)$$

where $dV = dy_1, dy_2 \ldots dy_p$

4. Let $\mu_k'(D_1^2)$ be the k-th raw moment of D_1^2, and let

$$M_k = \int_0^\infty D_1^{2k}\cdot e^{-\frac{np}{4}[D_1^2+\Delta^2-2D_1.\Delta\cos\theta]}.dV \qquad \dots \quad (4.1)$$

where the single sign of integration stands for the p-fold integration.

Then
$$\mu_k'(D_1^2) = C\cdot M_k \qquad \dots \quad (4.2)$$

we also write
$$np/4 = t \qquad \dots \quad (4.3)$$

and
$$M_k = \int_0^\infty D_1^{2k}\cdot e^{-t(D_1^2+\Delta^2-2D_1.\Delta\cos\theta)}\cdot dV \qquad \dots \quad (4.11)$$

Then

$$\frac{\partial M_k}{\partial\Delta} = -2t\int D_1^{2k}(\Delta-D_1\cos\theta)e^{-(tD_1^2+\Delta^2-2D_1\Delta\cos\theta.)}\cdot dV \qquad \dots \quad (4.4)$$

$$= -2t\cdot\Delta\cdot M_k+2t\int D_1^{2k+1}\cdot\cos\theta\cdot e^{-t(D_1^2+\Delta^2-2D_1\Delta\cos 0)}\cdot dV \qquad \dots \quad (4.41)$$

Again

$$\frac{\partial M_k}{\partial t} = -\int D_1^{2k}(D_1^2+\Delta^2-2D_1\cdot\Delta\cos\theta)e^{-t(D^2+\Delta^2-2D_1\Delta\cos 0)}\cdot dV \qquad \dots \quad (4.5)$$

$$= -M_{k+1}-\Delta^2\cdot M_k+\frac{\Delta}{t}\left[\frac{\partial M_k}{\partial\Delta}+2t\cdot\Delta\cdot M_k\right] \qquad \dots \quad (4.51)$$

Therefore

$$M_{k+1} = \Delta^2\cdot M_k+\frac{\Delta}{t}\cdot\frac{\partial M_k}{\partial\Delta}-\frac{\partial M_k}{\partial t} \qquad \dots \quad (4.6)$$

5. Now actually

$$M_0 = \int_0^\infty e^{-t[(y_1-\eta_1)^2+(y_2-\eta_2)^2+\dots(y_p-\eta_p)^2]}\cdot dy_1\, dy_2 \dots dy_p$$

$$= \left(\frac{\pi}{t}\right)^{p/2} \qquad \dots \quad (5.1)$$

Therefore

$$\frac{\partial M_0}{\partial t} = -\frac{p(\pi)^{p/2}}{2(t)^{(p+2)/2}} = \frac{p\cdot(\pi)^{p/2}}{2(t)^{(p+2)/2}} = \frac{p\cdot M_0}{2t} \qquad \dots \quad (5.2)$$

$$\frac{\partial M_0}{\partial\Delta} = 0 \qquad \dots \qquad \dots \qquad \dots \qquad \dots \quad (5.3)$$

Also $\mu_0'(D_1^2) = c \cdot M_0 = 1$ and therefore

$$c = \left(\frac{t}{\pi}\right)^{p/2} \qquad \dots \qquad \dots \qquad \dots \qquad \dots \quad (5.4)$$

We can use the above initial values to calculate $M_1, M_2, \dots M_k$ etc., and hence with the help of equation (4.2) obtain the raw moments (about any arbitrary origin) of D^2 which can be written in the form :—

$$\mu_k'(D_1^2) = \left(\frac{t}{\pi}\right)^{p/2} M_k \qquad \dots \qquad \dots \quad (5.5)$$

6. It will be convenient to use new variables

$$u = \Delta^2, \quad \text{and} \quad s = 1/t \qquad \dots \qquad \dots \quad (6.1)$$

Equation (4.6) can now be written as

$$M_{k+1} = u \cdot M_k + 4\, u {\cdot} s . \frac{\partial M_k}{\partial u} + 2s^2 \cdot \frac{\partial M_k}{\partial s} \qquad \dots \quad (6.2)$$

Remembering that $M_0 = (\pi \cdot s)^{p/2}$, we note from the form of equation (6.2) that M_k is a homogeneous function of order $(k+\frac{1}{2}p)$ in u and s.

Therefore

$$M_{k+1} = u {\cdot} M_k + 2u {\cdot} s {\cdot} \frac{\partial M_k}{\partial u} + 2s\left[u \frac{\partial M_k}{\partial u} + s \frac{\partial M_k}{\partial s}\right]$$

$$= u \cdot M_k + 2u \cdot s \cdot \frac{\partial M_k}{\partial u} + 2s(k+\tfrac{1}{2}p) {\cdot}\, M_k, \text{ by Euler's theorem} \qquad \dots \quad (6.3)$$

Thus

$$M_{k+1} = [s(2k+p)+u] M_k + 2u.s.\, \frac{\partial M_k}{\partial u} \qquad \dots \quad (6.4)$$

Writing briefly μ_k' for $\mu_k'(D_1^2)$, we have

$$\mu_k' = \frac{M_k}{(2s\pi)^{p/2}} \qquad \dots \quad (6.5)$$

Equation (6.4) may now be written in the form :—

$$\mu_{k+1}' = [s(2k+p)+u]\mu_k' + 2u \cdot s \cdot \frac{\partial \mu_k'}{\partial u} \qquad \dots \quad (6.6)$$

This is the fundamental recurrence formula. We notice that μ_k' is a homogeneous function of the k-th degree in s and u.

The general expression for μ_k' can be easily obtained.

$$\mu_k' = c_k \cdot u^k + c_{k-1} \cdot u^{k-1} \cdot s + \ldots c_0 \cdot s^k \qquad \ldots \quad (7.1)$$

Then from equation (6.6) we get

$$\mu_{k+1}' = C_k . u^{k+1} + [c_{k-1} + (p+4k) c_k] u^k \cdot s + \ldots + [c_{k-2} + (p+4k-2) c_{k-1}] u^{k-1} \cdot s^2 + \ldots$$
$$+ [c_0 + (p+2k+2) c_1] u . s^k + (p+2k) c_0 . s^{k+1} \qquad \ldots \quad (7.2)$$

From this equation we can immediately write down the coefficients occurring in μ_k' :—

μ_0'.........1
μ_1'.........1, p
μ_2'.........1, $2(p+2)$, $p(p+2)$,
μ_3'.........1, $3(p+4)$, $3(p+2)(p+4)$, $p(p+2)(p+4)$ etc.

The general coefficients can be easily proved by induction to be of the form :—

$$1 {}^k c_1 [p+2(k-1)],\ {}^k c_2 [\{p+2(k-1)\}\{p+2(k-2)\}], \ldots\ldots$$
$$\ldots\ldots {}^k c_r [\{p+2(k-1)\}\{p+2(k-2)\} \ldots\ldots \{p+2(k-r)\}],$$
$$\ldots {}^k c_k [\{p+2(k-1)\}\{p+2(k-2)\} \ldots p] \ldots\ldots \qquad \ldots \quad (7.4)$$

Remembering that $u = \Delta^2$, and $s = 1/2t = 2/(np)$ we finally obtain the general formula for the k-th raw moment-coefficient :—

$$\mu_k'(D_1^2) = \Delta^{2k} + \frac{2 . {}^k c_1 (p+2k-2)}{np} \cdot \Delta^{2k-2} + \frac{2^2 \cdot {}^k c_2 (p+2k-2)(p+2k-4)}{n^2 p^2} \cdot \Delta^{2k}$$
$$+ \ldots + \frac{2^k . (p+2k-2)(p+2k-4) \ldots (p+2)p}{n^k \cdot p^k} \qquad \ldots \quad (7.5)$$

$$= \underset{\alpha \to \infty}{\mathrm{Lt.}} \Delta^{2k} \cdot F \left(-k ; -k - \tfrac{1}{2} p + 1 ; \alpha ; \frac{4\alpha}{np\Delta^2} \right) \qquad \ldots \quad (7.6)$$

The two formulae (7.5) and (7.6) are in agreement with the results obtained by Rajchandra Bose[2] in equations (8.9) and (8.91) on *p*. 152 of *Sankhyā*, Vol. 2, Part 2.

8. It is clear from the preceding considerations that $\mu_k'(D^2)$'s, whose recurrence formulae have been calculated, belong to a Gaussian distribution in p-variables; it may be therefore interesting to deduce certain mathematical properties of these functions.

We start from the recurrence formula

$$\mu'_{k+1} = [s(2k+p)+u]\mu'_k + 2us\,\frac{\partial \mu'_k}{\partial s} \qquad \dots \quad (6.7)$$

Putting $\mu'_k = (2s)^k \cdot \psi_k(u/2s)$ and using $\chi = u/2s$ as the new variables, we have

$$\psi_{k+1}(x) = [(k+\tfrac{1}{2}p)+x]\psi_k(x) + x\psi'_k(x) \qquad \dots \quad (8.1)$$

Multiplying by $e^x.\ x^{k+\frac{1}{2}p-1}$, we have

$$e^x \cdot x^{k+\frac{1}{2}.-1}\psi_{k+1}(x) = \frac{d}{dx}\,[e^x \cdot x^{k+\frac{1}{2}p}\psi_k(x)],$$

$$\text{or} \quad e^x \cdot x^{k+\frac{1}{2}p+1}\psi_{k+1}(x) = x^2 \cdot \frac{d}{dx}\,[e^x \cdot x^{k+\frac{1}{2}p}\psi_k(x)] \qquad \dots \quad (8.2)$$

By regular descent, since $\psi_0(x) = 1$, it follows easily that

$$e^x \cdot x^{k+\frac{1}{2}p}\psi_k(x) = \left(x^2\,\frac{d}{dx}\right)^k [e^x \cdot x^{\frac{1}{2}p}] \qquad \dots \quad (8.3)$$

which may be regarded as a solution of the recurrence formula.

If we put $t = 1/x = 2s/u$, we get,

$$\mu'_k = (2s)^k.\ \psi_k(x) = (2st)^k.\ e^{-1/t}\ .\ t^{\frac{1}{2}p}\left(-\frac{d}{dt}\right)^k [e^{1/t} \cdot t^{-\frac{1}{2}p}] \qquad \dots \quad (8.4)$$

which is identical with (8.6) in Rajchandra Bose's paper.[3]

Starting from the equation (8.3) and denoting by y, the expression $e^x x^{\frac{1}{2}p}$, and by D, the operator $x^2\,\dfrac{d}{dx}$, we easily find that y satisfies the differential equation

$$\frac{1}{x^2}\,Dy = \left(\tfrac{1}{2}p \cdot \frac{1}{x} + 1\right)y.$$

Operating on both sides of this equation by D^{k+1} and remembering that $D^k y = \psi_k(x)$ one easily finds that $\psi_k(x)$ satisfies the differential equation

$$x\,\frac{d^2\psi_k}{dx^2} + (\tfrac{1}{2}p+x)\,\frac{d\psi_k}{dx} - k\psi_k = 0 \qquad \dots \quad (8.5)$$

If $\qquad \psi_k(-x) = (-1)^k \cdot f_k(x),$

$f_k(x)$ satisfies the differential equation

$$x\,\frac{d^2 f_k}{dx^2} + (\tfrac{1}{2}p-x)\,\frac{df_k}{dx} + k \cdot f_k = 0 \qquad \dots \quad (8.6)$$

The solutions are easily seen to be polynomials of order k, when k is an integer, and these in case of (8.6) have been studied by Sonine, Gegenbauer, and others. The differential equations have also been studied by Whittaker and others, and give rise to solutions of the confluent hypergeometric types.

9. There is an alternative method of arriving at the differential equation for μ', which is here appended as the method is capable of wide generalisation.

The Gaussian distribution function

$$F = \frac{1}{(2\pi t)^{\frac{1}{2}p}} \cdot e^{-\Sigma(x_r..y_r)^2/2t}, \text{ where } r = 1, 2, 3, \ldots p$$

can be easily seen to satisfy the differential equation

$$\frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} + \ldots + \frac{\partial^2 F}{\partial x_p^2} = 2\frac{\partial F}{\partial t} \quad \ldots \quad (9.1)$$

where instead of $(x_1, x_2, \ldots x_p)$, we may also take $(y_1, y_2 \ldots y_p)$ as variables.

The expression $\frac{1}{(2\pi t)^{\frac{1}{2}p}} \cdot e^{-\Sigma_r(x_r-y_r)^2/2t}$ denoted by F also satisfies the differential equation

$$\frac{\partial^2 F}{\partial y_1^2} + \frac{\partial^2 F}{\partial y_2^2} + \ldots\ldots + \frac{\partial^2 F}{\partial y_p} = 2\frac{\partial F}{\partial t} \quad \ldots \quad (9.11)$$

Now, since this differential equation is linear which means that if $\psi_1, \psi_2, \ldots$ etc. are each a solution of the differential equation then $\psi_1+\psi_2+\ldots$ etc. is a solution of the same equation, and since an integral is nothing but a sum, therefore,

$$\psi(y_1, y_2, \ldots y_p, t) \equiv \int \phi(x_1, x_2, \ldots x_p) \cdot \frac{1}{(2\pi t)^{\frac{1}{2}p}} \cdot e^{-\Sigma(x_r-y_r)^2/2t} dx_1, dx_2 \ldots dx_p) \quad \ldots \quad (9.2)$$

which consists of a sum of terms like $\frac{1}{(2\pi t)^{\frac{1}{2}p}} \cdot e^{-\Sigma(x_r-y_r)^2/2t}$ each satisfying the differential equation and each multiplied by a term like $\phi(x_1, x_2 \ldots x_p) \cdot dx_1\, dx_2 \ldots dx_p$ not depending upon the independent variables the y's, is easily seen to satisfy the equation (9.11).

Therefore

$$\mu_k'(y_1, y_2, \ldots y_p, t) = \int (x_1^2+x_2^2+\ldots x_p^2)^k \cdot \frac{1}{(2\pi t)^{\frac{1}{2}p}} \cdot e^{-\Sigma(x_r-y_r)^2/2t} \times dx_1 . dx_2 \ldots dx_p \quad (9.3)$$

also satisfies the same differential equation.

An orthogonal transformation of (x's) and (y's) will not change the integrand; also making a transformation

$$\left.\begin{array}{l} x_1' = \lambda x_1',\ x_2 = \lambda x_2', \quad \ldots\ldots\ldots\ldots t = \lambda^2 t' \\ y_1 = \lambda y_1',\ y_2 = \lambda y_2', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad \ldots \quad (9.4)$$

multiplies the integral by λ^{2k}. Hence

$$\mu_k'(y_1, y_2, \ldots y_p, t) = \mu_k'[y_1^2+y_2^2+\cdots y_p^2, t] \qquad \ldots \quad (9.41)$$

Putting $$y_1^2+y_2^2+\ldots y_p^2 = R^2 \qquad \ldots \quad (9.42)$$

it follows that $$\mu_k'(y_1, y_2, \ldots y_p, t) = \mu_k'(R^2, t) \qquad \ldots \quad (9.5)$$

also $$\mu_k'(\lambda R^2, \lambda t) = \lambda^k \mu_k'(R^2, t) \qquad \ldots \quad (9.51)$$

or μ_k' is a homogeneous function of (R^2, t) of the kth degree. Therefore μ_k' is a homogeneous solution of the kth degree of (R^2, t) alone of the equation (9.1). Making a change of variables to R and angle-coordinates in a p-dimensional space, μ_k' is seen to satisfy the differential equation

$$\frac{1}{R^{p-1}} \cdot \frac{\partial}{\partial R}\left[R^{p-1}\frac{\partial \mu_k'}{\partial R}\right] = 2\frac{\partial \mu_k'}{\partial t} \qquad \ldots \quad (9.6)$$

This transformation is seen easily as $\Box_p^2 \cdot F =$ Div. grad F, and when F is a function of R alone, the expression follows as the hyper-spherical element perpendicular to R is proportional to R^{p-1}. Thus

$$\Box_p^2 \cdot F = \frac{1}{R^{p-1}} \cdot \frac{\partial}{\partial R}\left[R^{p-1} \cdot \frac{\partial F}{\partial R}\right] = \frac{\partial^2 \mu_k'}{\partial R^2} + \frac{p-1}{R} \cdot \frac{\partial \mu_k'}{\partial R} = 2\,\frac{\partial \mu_k'}{\partial t} \qquad \ldots \quad (9.7)$$

Finally choosing $R^2 = 2t \cdot x,\ \mu_k' = (2t)^k \cdot \psi_k(x)$ $\qquad \ldots \quad (9.8)$

the differential equation of $\psi_k(x)$ is easily seen to be

$$x\frac{d^2\psi_k}{dx^2} + (\tfrac{1}{2}p + x)\frac{d\psi_k}{dx} - k\psi_k = 0 \qquad \ldots \quad (9.9)$$

which is identical with the equation (8.5)

10. The method sketched above enables one to calculate the moment functions even when k is not an integer; actually however, unless k's are integers, μ_k''s will not be expressed by a polynomial but by an infinite series. The differential equation itself determines all the constants except one, which can be determined from simple considerations.

I will conclude my remarks by deducing certain properties of the general differential equation, which will be useful for determining the form of the general distribution function about which I wish to make some remarks in future.

Consider the equation

$$\Box_p^2 . F = \frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} + \dots \frac{\partial^2 F}{\partial x_p^2} \doteq 2 \frac{\partial F}{\partial t} \qquad \dots \quad (10.1)$$

Suppose ψ is a solution of the differential equation, we then seek a solution of the equation of the form $F = \psi . \chi$.
Then

$$\psi . (\Box_p^2 . \chi) = \chi . (\Box_p^2 . \psi) + 2(D\psi, D\chi) = 2 \left[\psi \frac{\partial \chi}{\partial t} + \chi . \frac{\partial \psi}{\partial t} \right] \qquad \dots \quad (10.2)$$

or
$$\psi(\Box_p^2 . \chi) + 2(D\psi, D\chi) = 2\psi \frac{\partial \chi}{\partial t} \qquad \dots \quad (10.3)$$

Now if
$$\psi = \frac{1}{(2\pi t)^{\frac{1}{2}p}} . e^{-\Sigma x_r^2/2t}, \text{ then } \frac{\partial \psi}{\partial x_r} = -\frac{x_r}{t} . \psi \qquad \dots \quad (10.4)$$

Therefore χ satisfies the differential equation

$$\Box_p^2 . \chi = \frac{2}{t} \left[x_1 \frac{\partial \chi}{\partial x_1} + x_2 \frac{\partial \chi}{\partial x_2} + \dots + t \frac{\partial \chi}{\partial t} \right] \qquad \dots \quad (10.5)$$

Suppose $F(x_1, x_2, \dots x_p, t)$ is a solution of the equation

$$\Box_p^2 . F = 2 \frac{\partial F}{\partial t} \qquad \dots \quad (10.6)$$

Then
$$\psi(x_1, x_2, \dots x_p, t) = F(x_1/t, x_2/t, \dots x_p/t, -1/t) \qquad \dots \quad (10.7)$$

will satisfy the differential equation (10.5), for putting $x'_r = x_r/t$, and $t' = -1/t$, it follows that

$$\frac{\partial \psi}{\partial x_r} = -t' \frac{\partial F}{\partial x'_r}, \qquad \dots \quad (10.8)$$

$$t \frac{\partial \psi}{\partial t} + \Sigma x_r \frac{\partial \psi}{\partial x_r} = -t' \frac{\partial F}{\partial t'} \qquad \dots \quad (10.81)$$

$$\Box_p^2 . \psi - \frac{2}{t}\left[t\frac{\partial\psi}{\partial t} + \dots\right] = t'^2\left[\Box_p'^2 . F - 2\frac{\partial F}{\partial t'}\right] = 0 \qquad \dots \quad (10.82)$$

11. This theorem enables us to construct solutions of the fundamental differential equation of the form

$$\frac{1}{(2\pi t)^{\frac{1}{2}p}} . e^{-\Sigma x_r^2/2t} . H\left(\frac{x_1}{t}, \frac{x_2}{t}, \dots \frac{x_p}{t}, -\frac{1}{t}\right) \qquad \dots \quad (11.1)$$

where $H(x_1, x_2, \dots x_p, t)$ is a solution of the same differential equation (9.1).

We have seen that $\mu_k'(R^2, t)$ the moment functions, are solutions of the differential equation (9.1)

$$\mu_k' = (2t)^k . \psi_k(R^2/2t) = 2t^k . \psi_k(y) \qquad \dots \quad (11.2)$$

where $\psi_k(y)$ satisfies the differential equation (8.4). Therefore,

$$\mu_k'\left(\frac{x_1}{t}, \frac{x_2}{t}, \dots \frac{x}{t}, -\frac{1}{t}\right) = (-1)^k . \frac{2^k}{t^k} . \psi_k\left[\frac{x_1^2 + x_2^2 + \dots x^2}{2t^2(-1/t)}\right]$$

$$= (-1)^k . (2/t)^k . \psi_k(-y) \qquad \dots \quad (11.3)$$

where $(-1)^k . \psi_k(-y) = f(y)$ which satisfies the equation (8.5).

If $y = \Sigma x_1^2/2t$, and $(-1)^k . \psi_k(-y)$ are the polynomials of Sonine-Gegenbauer, it follows that

$$\frac{1}{(2\pi t)^{\frac{1}{2}p}} . e^{-y} . (-1)^k . (2/t)^k . \psi_k(-y) \qquad \text{are solutions of (9.1).}$$

12. A distribution function which satisfies the fundamental equation, and is of a certain type can be written in the form

$$\frac{1}{(2\pi t)^{\frac{1}{2}p}} . e^{-y} . \left[1 - C_1 . \frac{\psi_1(-y)}{t} + \dots (-1)^k . C_k . \frac{\psi_k(-y)}{t^k} + \dots\right] \text{where } y = \Sigma x_r^2/2t$$

Certain orthogonal properties of the Sonine polynomials will be useful to recall here. The functions $S_k(x) = (-1)^k . C_k . \psi_k(-x)$ satisfies, we have seen the differential equation

$$x\frac{d^2 S_k}{dx^2} + (\tfrac{1}{2}p - x)\frac{dS_k}{dx} + k . S_k = 0 \qquad \dots \quad (8.5)$$

which can be written in the self-adjoint form,

$$\frac{d}{dx}\left(x^{\frac{1}{2}p}.e^{-x}.\ \frac{dS_k}{dx}\right)+k.x^{\frac{1}{2}p-1}.\ e^{-x}S_k=0 \qquad \dots \quad (12.2)$$

from which the orthogonal properties follow at once.

Taking two polynomials of orders k and k', we have

$$(k-k')\int_0^\infty e^{-x}.\ x^{\frac{1}{2}p-1},\ S_k.\ S_{k'}\ dx=0 \qquad \dots \quad (12.3)$$

Hence if $$k\neq k',\ \int_0^\infty e^{-x}.\ x^{\frac{1}{2}p-1}.\ S_k.\ S_{k'}.\ dx=0 \qquad \dots \quad (12.31)$$

also $$\int_0^\infty e^{-x}\cdot x^{\frac{1}{2}p-1}.\ S_k^2(x)dx\neq 0=A=1 \qquad \dots \quad (12.32)$$

since the arbitrary constant in the definition of the polynomial may be so adjusted as to make $A=1$, which we may call the normalised polynomials.

Suppose the distribution functions to be $\phi(x_1, x_2 \dots x_p, t)$ then the distribution function in R^2 is obviously

$$\int_R^{R+dR} \phi(x_1, x_2 \dots x_p, t)dx_1 \cdot dx_2 \dots dx_p \qquad \dots \quad (12.4)$$

taken between the hyper-spheres of radii R and $(R+dR)$.

It will take the form

$$A\cdot\psi(R^2, t)\cdot R^{p-1}\cdot dR=\psi'(D, t)\cdot D^{\frac{1}{2}p-1}\cdot dD \text{ when } D=R^2. \qquad \dots \quad (12.41)$$

We shall call $\psi'(D, t)$ the distribution function, when the number of variables is p.

Then $$\int_0^\infty \psi'(D, t)\cdot D^{\frac{1}{2}p-1}\cdot dD=1 \qquad \dots \quad (12.5)$$

$$\int_0^\infty D^k\cdot\psi'(D, t)\cdot D^{\frac{1}{2}p-1}\cdot dD=\mu_k(t) \qquad \dots \quad (12.51)$$

It is also clear that if $y=D/2t$, then substituting the expansion of $S_k\ (y)$ which can be easily obtained from the differential equation or from the expression for μ_k' found before, we get

$$\int_0^\infty (2t)^k \cdot S_k(y) \cdot \psi'(D,t) \cdot D^{\frac{1}{2}p-1} \cdot dD$$

$$= C_k \int_0^\infty \left[D^k - k(\tfrac{1}{2}p+k-1)\cdot D^{k-1}\cdot (2t) + \frac{k(k-1)}{2!}(\tfrac{1}{2}p+k-1)(\tfrac{1}{2}p+k-2)\cdot D^{k-2}\cdot (2t)^2 + \dots\right] \times \psi(D,t)\cdot D^{\frac{1}{2}p-1}\cdot dD \qquad \dots \quad (12.6)$$

$$= C_k\left[\mu'_k - \frac{k(\frac{1}{2}p+k-1)}{1!}\mu'_{k-1}\cdot (2t) + \text{etc} \dots\right] \qquad \dots \quad (12.61)$$

This expression is thus calculable in terms of the moments of the different orders up to k.

Assuming now

$$\psi(D,t) = \frac{1}{(2\pi t)^{\frac{1}{2}p}}\cdot e^{-y}\left[1 + \frac{A_1}{t}S_1(y) + \frac{A_2}{t^2}S_2(y) + \dots + \frac{A_k}{t^k}\cdot S_k(y) + \dots\right] \qquad \dots \quad (12.7)$$

and remembering that the above series is so chosen that each term, and therefore the whole series, if convergent, satisfies the differential equation $\Box_p^2 F = 2\partial F/\partial t$, we have:

$$\int \psi(D,t)\cdot D^{\frac{1}{2}p-1}\cdot (2t)^k S_k(y)\cdot dD$$

$$= \int \frac{1}{(2\pi t)^{\frac{1}{2}p}}\cdot e^{-y}\left[1 + \frac{A}{t}S_1(y) + \dots \frac{A_k}{t_k}\cdot S_k(y) + \dots\right](2t)^k S_k(y)(2t)^{p}\cdot y^{\frac{1}{2}p-1}\cdot dy$$

$$= \frac{1}{(2\pi t)^{\frac{1}{2}p}}\cdot 2^k A_k$$

$$= C_k\left[\mu'_k - \frac{k(\frac{1}{2}p+k-1)}{1!}\mu'_{k-1}(2t) + \frac{k(k-1)}{2!}(\tfrac{1}{2}p+k-1)(\tfrac{1}{2}p+k-2)\mu'_{k-2}(2t)^2 + \dots\right] \quad (12.8)$$

From (12.6) since $y = D/2t$, we get finally

$$A_k = \frac{(2\pi)^{\frac{1}{2}p}}{2^k}\cdot C_k\cdot\left[\mu'_k - \frac{k(\frac{1}{2}p+k-1)}{1!}\mu'_{k-1} + \dots\right] \qquad \dots \quad (12.9)$$

which determines the constants A_k in the distribution formula in terms of the moments. Hence the distribution function can be calculated in terms of the moments. Rajchandra Bose's distribution function obviously can be put in the form indicated above, the actual calculation would give interesting integral relations involving the Bessel functions.

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ON THE MOMENT-COEFFICIENTS OF THE D^2-STATISTIC AND CERTAIN INTEGRAL AND DIFFERENTIAL EQUATIONS CONNECTED WITH THE MULTIVARIATE NORMAL POPULATION

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INTRODUCTION.

In a previous paper[1] I have considered certain recursion formulæ and relations between the moment-coefficients of the D^2-statistic. Starting with certain further algebraic identities between the moment-coefficients of different orders of the D^2-statistic, I reach a number of fundamental integral and differential equations connected with the multivariate normal distribution. A number of solutions of these equations are discussed in the present paper.

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[1] "On the Complete Moment-Coefficients of the D^2-statistic." *Sankhyā*, Vol. 2 (4), 1936, 385-396.

ALGEBRAIC IDENTITIES DEDUCED FROM THE POLYNOMIAL FORM OF THE MOMENT-COEFFICIENTS OF THE D^2-STATISTIC.

1. I start with the identities that exist between the moment-functions of the different orders of the D^2-statistic, which can be regarded as the square modulus statistic for a Gaussian distribution in p-variates.

It has been already proved[2] that

$$\mu_k'(l^2) = \int (x_1^2+x_2^2+\ldots+x_p^2)^k \cdot \frac{e^{-\Sigma_r(x_r-y_r)^2/2t}}{(2\pi t)^{p/2}} \cdot dV \qquad \ldots \quad (1.1)$$

where $$x_1^2+x_2^2+\ldots x_p^2 = l^2 \qquad \ldots \quad (1.11)$$

and $$y_1^2+y_2^2+\ldots y_p^2 = \Delta^2 \quad (\text{or } u) \qquad \ldots \quad (1.12)$$

and μ_k is a homogeneous function of $(\Delta^2,\ 2t)$ of the k-th degree of the form[3] :

$$\mu_k(l^2) = \beta^k + k(k+p/2-1)\cdot\beta^{k-1}(2t)+\ldots$$
$$+\frac{\Gamma(k+1)\cdot\Gamma(k+p/2)\cdot\beta^{k-r}(2t)^r}{\Gamma(r+1)\cdot\Gamma(k+1-r)\cdot\Gamma(k+p/2-r)}+\ldots+\frac{\Gamma(k+p/2)}{\Gamma(p/2)}\cdot(2t)^k \qquad \ldots \quad (1.2)$$

It should be noted here that in (1·2) and in subsequent equations we are writing μ_k for μ_k' and β for Δ^2 for convenience of printing.

It is easily seen by direct algebraical methods that

$$\beta^k = \mu_k - k(k+p/2-1)\mu_{k-1}\cdot(2t)+\ldots+(-1)^r\cdot\mu_{k-r}\cdot(2t)^r\frac{\Gamma(k+1)\Gamma(k+p/2)}{\Gamma(r+1)\,\Gamma(k+1-r)\,\Gamma(k+p/2-r)}$$
$$+\ldots+(-1)^k\cdot\frac{\Gamma(k+p/2)}{\Gamma(p/2)}\cdot(2t)^k \equiv B_k \qquad \ldots \quad (1.3)$$

So that the various identical relations between μ's involving p and t, which will be called the identities of the first type, can be written in the form $(B_k)^{k'} = (B_{k'})^k$, or symbolically, $(k\ k') = 0$. These identities can be written down by taking any two expressions B_k and $B_{k'}$ and their existence is self-evident from their form.

Also since B_k's are polynomials in p or in $(2t)$, by taking any three expressions $\beta^a = B_a$, $\beta^b = B_b$, and $\beta^c = B_c$, we can eliminate (β, p) or (β, t) and thus arrive at identities involving alternatively either μ's and t's or μ's and p's. These identities can be written as

$$[abc]_p = 0, \quad [abc]_t = 0 \text{ etc.} \qquad \ldots \quad (1.4)$$

where the suffix denotes the variable which has been eliminated in the process.

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[2] *Sankhyā*, 2(4), 1936, p. 392, equation (9).

[3] *Sankhyā*, 2(4), 1936, p. 389, equation (7.5).

Finally by taking any four of the expressions we can eliminate β, p and t, arrive at relations between μ's alone, which can be written as

$$\{a\ b\ c\ d\} = 0 \qquad \ldots \quad (1.5)$$

These idential relations are interesting, and may be regarded as characteristic of the modulus distribution of the Gaussian type.

Direct algebraical computation will easily verify the following typical relations :—

$$2pt^2 - 4t\mu_1 + (\mu_2 - \mu_1^2) = 0 \qquad \ldots \quad (1.51)$$

$$[123]_p = 8\mu_1 t + 8t(\mu_1^2 - \mu_2) + (\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3) \qquad \ldots \quad (1.52)$$

The expression (1234) = 0 is more complicated; by taking B_1, B_2, B_3, B_4, and putting

$$\left.\begin{aligned} &A_2 = \mu_1^2 - \mu_2, \quad A_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \\ \text{and} \qquad &A_4 = \mu_4 - 4\mu_2\mu_3 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 6\mu_1^4 \end{aligned}\right\} \qquad \ldots \quad (1.61)$$

the existence of the following two quadratic equations can be proved,

$$\left.\begin{aligned} &8\mu_1 t^2 + 8A_2 t + A_3 = 0 \\ \text{and} \qquad &A_2 t^2 + A_3 t - A_4/12 = 0 \end{aligned}\right\} \qquad \ldots \quad (1.62)$$

from which the bracket (1234) = 0 follows as the t-eliminant of the equations.

ALGEBRAIC IDENTITIES DEDUCED FROM THE DIFFERENTIAL FORM OF THE MOMENT-COEFFICIENT.

2. It is satisfactory to note here that the identities of the above type, which can be arrived at by laborious algebraical methods, can be deduced much more simply by the following analytical procedure, which furnishes the *raison d'être* of identities of the above types, and especially of the existence of the quadratic equations. This method also allows us to investigate the inverse problem, namely, the nature of the distribution function for the modulus when the identities of the above kind are postulated for the moment-functions.

We start from the differential form of the moment-coefficient deduced before[4] for the D^2-statistic :—

$$\mu_k = (2st)^k \cdot e^{-1/t} \cdot t^{p/2} \cdot \left(-\frac{d}{dt}\right)^k (e^{1/t} \cdot t^{-p/2}) \qquad \ldots \quad (2.1)$$

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[4] *Sankhyā*, 2(4), 1936, p. 390, equation (8.4).

where $t = 2s/u$ according to the present notation.

Then it follows from the fact that μ_k is a homogeneous function of $(2s, u)$ of the k-th degree, that

$$\frac{\mu_k}{(2st)^k} = \frac{\mu_k}{(ut^2)^k} = \frac{\mu_k}{u^k t^{2k}} = \psi_k(t) = e^{-1/t}\cdot t^{p/2}\cdot\left(\frac{-d}{dt}\right)^k (e^{1/t}\cdot t^{-p/2}) \quad \ldots \quad (2.2)$$

So that $\psi(t)$ is a polynomial of $-k$-th degree in t alone. Also the differential form of $\psi(t)$ can be transformed into an integral form. Assuming that

$$\frac{e^{1/t}}{t^{p/2}} = Z(t) = \int_0^\infty e^{-xt}\cdot\phi(x)\cdot dx \quad \ldots \quad (2.3)$$

where $\phi(x)$ is a function whose properties we shall investigate later, it follows that

$$\psi_k(t) = \frac{1}{Z(t)}\cdot\left(-\frac{d}{dt}\right)^k \int_0^\infty e^{-xt}\cdot\phi(x)\cdot dx \quad \ldots \quad (2.31)$$

which can be written in the form :—

$$\psi_k = \frac{\int_0^\infty x^k\cdot e^{-xt}\cdot\phi(x)\cdot dx}{\int_0^\infty e^{-xt}\cdot\phi(x)\cdot dx} \quad \ldots \quad (2.32)$$

From this it follows that

$$\psi_k(t)\cdot\int_0^\infty e^{-xt}\cdot\phi(x)\cdot dx = \int_0^\infty x^k\cdot e^{-xt}\cdot\phi(x)\cdot dx \quad \ldots \quad (2.41)$$

Hence by differentiation with regard to t, we have

$$\frac{d\psi_k}{dt}\cdot\int_0^\infty e^{-xt}\cdot\phi(x)\cdot dx - \psi_k\cdot\int_0^\infty xe^{-xt}\cdot\phi(x)\cdot dx = -\int_0^\infty x^{k+1}\cdot e^{-xt}\cdot\phi(x)\cdot dx \ldots \quad (2.42)$$

Remembering the general form of ψ_k in (2.31) it follows that

$$\frac{d\psi_k}{dt} = \psi_1\cdot\psi_k - \psi_{k+1} \quad \ldots \quad (2.43)$$

This is fundamental relation which allows us to calculate the various differential coefficients of ψ's in terms of ψ's of different orders. Thus

$$\left.\begin{aligned} \frac{d\psi_1}{dt} &= \psi_1^2 - \psi_2, \quad \frac{d^2\psi_1}{dt^2} = 2\psi_1^3 - 3\psi_1\psi_2 + \psi_3 \\ \frac{d^3\psi_1}{dt^3} &= 6\psi_1^4 - 12\psi_1^2\psi_2 + 3\psi_2^2 + 4\psi_1\psi_3 - \psi_4 \end{aligned}\right\} \quad \ldots \quad (2.51)$$

Since $(\psi_1 t^2)$ is a linear function of t it follows that the following differential relations are true.

$$\frac{d^2(\psi_1 t^2)}{dt^2} = 0, \ \frac{d^3(\psi_1 t^2)}{dt^3} = 0, \ \frac{d^4(\psi_1 t^2)}{dt^4} = 0 \quad \ldots \quad (2.52)$$

$$\left.\begin{aligned} t^2\cdot \frac{d^2\psi_1}{dt^2} + 4t\cdot \frac{d\psi_1}{dt} + 2\psi_1 &= 0 \\ t^2\cdot \frac{d^3\psi_1}{dt^3} + 6t\cdot \frac{d^2\psi_1}{dt^2} + 3\,\frac{d\psi_1}{dt} &= 0 \end{aligned}\right\} \quad \ldots \quad (2.61)$$

and generally $$t^2\cdot \frac{d^k\psi_1}{dt^k} + 2kt\cdot \frac{d^{k-1}\psi_1}{dt^{k-1}} + k(k-1)\cdot \frac{d^{k-2}\psi_1}{dt^{k-2}} = 0 \quad \ldots \quad (2.62)$$

If we now remember that the differential coefficients of ψ_1's are expressible in ψ^k's of different orders, and that ψ's are only μ's divided by powers of t, the existence of the quadratic equations of the type found by algebraical method is evident. This at the same time allows us to write down the equations with much less labour.

Using equations (2.51) and equation (2.2), we have

$$\left.\begin{aligned} t^2(2\psi_1^3-3\psi_1\psi_2+\psi_3)+4t(\psi_1^2-\psi_2)+2\psi_1 &= 0 \\ t^2(6\psi_1^4-12\psi_1^2\psi_2+3\psi_2^2+4\psi_1\psi_3-\psi_4)+6t(2\psi_1^2-3\psi_1\psi_2+\psi_3)+3(\psi_2^2-\psi_2) &= 0 \end{aligned}\right\} \ldots \quad (2.63)$$

Now substituting the values of ψ's in terms of μ's the above two equations reduce respectively to (2.65) and (2.66) given below

$$\frac{t^2(2\mu_1^3-3\mu_1\mu_2+\mu_3)}{u^3t^6} + \frac{4t(\mu_1^2-\mu_2)}{u^2t^4} + \frac{2\mu_1}{ut^2} = 0 \quad \ldots \quad (2.64)$$

which can be written, by substituting $2s/u$ for t, in the form

$$8\mu_1 s^2+8(\mu_1^2-\mu_2)s+(2\mu_1^2-3\mu_1\mu_2+\mu_3)=0 \qquad \ldots \quad (2.65)$$

and $(\mu_1^2-\mu_2^2)s^2+(2\mu_1^3-3\mu_1\mu_2+\mu_3)s+(1/12)(6\mu_1^4-12\mu_1^2\mu_3+3\mu_2^2+4\mu_1\mu_3-\mu_4)$ .. (2.66)

These are the quadratic equations arrived at by direct elimination and referred to in (1.62)

The fundamental { } expression alluded to before in (1.5) can also be written in a form involving the differential co-efficients of ψ_1 's of different orders.

Thus writing
$$\psi_1^k \equiv \frac{d^k\,\psi_1}{dt^k} \qquad \ldots \quad (2.7)$$

we have also two relations like

$$\frac{d^k\,(\psi_1 t^2)}{dt}=0, \quad \frac{d^\sigma\,(\psi_1 t^2)}{dt^\sigma}=0 \qquad \ldots \quad (2.81)$$

which can be written in the form

$$\left.\begin{array}{l} \psi_1^k \cdot t^2+2k\cdot\psi_1^{k-1}\cdot t+k(k-1)\psi_1^{k-1}=0 \\ \psi_1^\sigma\cdot t^2+2\sigma\cdot\psi_1^{\sigma-1}\cdot t+\sigma(\sigma-1)\psi_1^{\sigma-2}=0 \end{array}\right\} \qquad \ldots \quad (2.82)$$

Hence the following relation which is equivalent to (1.5) can be arrived at as the t-eliminant of the above two quadratic equations :—

$$[k(k-1)\psi_1^\sigma\cdot\psi_1^{k-2}-\sigma(\sigma-1)\psi_1^{\sigma-2}\cdot\psi_1^k]^2$$
$$=4[k\sigma(\sigma-1)\psi_1^{k-1}\cdot\psi_1^{\sigma-2}-\sigma k(\sigma-1)\psi_1^{\sigma-1}\cdot\psi_1^{k-2}][\sigma\cdot\psi_1^k\psi_1^{\sigma-1}-k\psi_1^\sigma\cdot\psi_1^{k-1}] \ldots \quad (2.9)$$

There is however one fundamental relation. The other relations deduced should therefore reduce to the fundamental one calculated above, where the different relations between μ's of different orders are taken into account.

THE INTEGRAL EQUATION CONNECTED WITH THE MOMENT-COEFFICIENTS AND ITS SOLUTIONS.

3. I shall now take up the integral form of $\psi_k(t)$

$$\psi_k(t)=\frac{\int\limits_0^\infty x^k\cdot e^{-xt}\,.\,\phi(x)\cdot dx}{\int\limits_0^\infty e^{-xt}\cdot\phi(x)\cdot dx} \qquad \ldots \quad (2.32)$$

The form shows that $\psi_k(t)$'s are moment-functions of a certain x-distribution and a knowledge of $\phi(x)$ will determine the distribution function to which $\psi(t)$'s can be related. In other words $\phi(x)$ will determine the μ_k's with which $\psi_k(t)$'s are simply connected. The existence of the algebraic identities between the μ's follows from two separate hypotheses—(1) the existence of the above integral relation, and (2) the linear form of $\psi_1(t)\cdot(t)^2$. We can therefore see what form of $\phi(x)$ (which for the present discussion can be regarded as arbitrary at first) is determined by the hypothesis that $\psi_1(t)\cdot t^2$ is a linear function of t.

Since

$$\psi_1(t) = \frac{\int\limits_0^\infty x\cdot e^{-xt}\cdot\phi(x)\cdot dx}{\int\limits_0^\infty e^{-xt}\cdot\phi(x)\cdot dx} \qquad \ldots \quad (3.10)$$

That is,

$$\psi_1(t)\cdot\int\limits_0^\infty e^{-xt}\cdot\phi(x)\cdot dx = \int\limits_0^\infty x\cdot e^{-xt}\cdot\phi(x)\cdot dx \qquad \ldots \quad (3.11)$$

The assumption of the linear form of $\psi_1(t)\cdot t^2$ implies that

$$(At+B)\int\limits_0^\infty e^{-xt}\cdot\phi(x)\cdot dx = \int\limits_0^\infty x\cdot t^2\cdot e^{-tx}\cdot\phi(x)\cdot dx \qquad \ldots \quad (3.12)$$

or

$$\int\limits_0^\infty e^{-xt}\,[x\cdot\phi(x)\cdot t^2 - A\cdot t\cdot\phi(x) - B\cdot\phi(x)]\cdot dx = 0 \qquad \ldots \quad (3.13)$$

The above relation can be transformed by integration by parts. Thus

$$\int\limits_0^\infty e^{-xt}\cdot x\cdot\phi(x)\cdot t^2\cdot dx = \left[-t\cdot x\cdot\phi(x)\cdot e^{-xt} - \frac{d}{dx}\{x\phi(x)\}\cdot e^{-xt}\right]_0^\infty + \int\frac{d^2}{dx^2}\{x\phi(x)\}\cdot e^{-xt}\cdot dx \qquad \ldots \quad (3.21)$$

Similarly

$$\int\limits_0^\infty t\cdot e^{-xt}\cdot\phi(x)\cdot dx = \left[-\phi(x)\cdot e^{-xt}\right]_0^\infty + \int\limits_0^\infty e^{-xt}\cdot\frac{d\phi}{dx}\cdot dx \qquad \ldots \quad (3.22)$$

The relation (3.13) implies

$$\left[-t\cdot x\cdot\phi(x)\cdot e^{-xt} - \frac{d}{dx}\{x\phi(x)\}\cdot e^{-xt} + A\cdot\phi(x)\cdot e^{-xt}\right]_0^\infty + \int\limits_0^\infty e^{-xt}\left\{\frac{d^2}{dx^2}\{x\phi(x)\} - A\cdot\frac{d\phi(x)}{dx} - B\cdot\phi(x)\right\}dx = 0 \qquad \ldots \quad (3.23)$$

Thus

$$\left\{e^{-xt}\left\{-x\frac{d\phi}{dx}-(tx+1-A)\phi(x)\right\}\right\}_0^\infty+\int_0^\infty e^{-xt}\left\{\frac{d^2}{dx^2}\{x\phi(x)\}-A\cdot\frac{d\phi}{dx}-B\phi\right\}=0 \qquad \ldots \quad (3.24)$$

This shows that $\phi(x)$ is a regular solution existing up to infinity of the following differential equation :

$$\frac{d^2}{dx^2}[x\cdot\phi(x)]-A\cdot\frac{d\phi(x)}{dx}-B\cdot\phi(x)=0 \qquad \ldots \quad (3.25)$$

if $\left(\frac{d\phi}{dx}\right)$ is finite and either $A=1$, or $\phi(x)=0$, when $x=0$.

The differential equation

$$\frac{d^2}{dx^2}(x\cdot\phi)-A\cdot\frac{d\phi}{dx}-B\phi\cdot(x)=0 \qquad \ldots \quad (3.25)$$

can be transformed by the substitution

$$x=R^2/\rho \qquad \ldots \quad (3.26)$$

So that

$$dx=\frac{2R\cdot dR}{\rho},\ \text{and}\ \frac{d}{dx}=\frac{\rho}{2R}\cdot\frac{d}{dR} \qquad \ldots \quad (3.27)$$

Hence the equation becomes

$$\frac{d^2\phi}{dR^2}-\frac{(2A-3)}{R}\cdot\frac{d\phi}{dR}-\frac{4B\cdot\phi}{\rho}=0 \qquad \ldots \quad (3.28)$$

The different cases for different positive and negative values of A and B can be easily seen to be solvable in terms of the functions of the type $J_\lambda(r)$ or $J_\lambda(ir)$.

When $A=p/2$, $B=1$, the case reduces to that of the D^2-statistic. The differential equation then becomes

$$\frac{d^2\phi}{dR^2}-\frac{(p-3)}{R}\cdot\frac{d\phi}{dR}-\phi=0,\ \text{when}\ \rho=4 \qquad \ldots \quad (3.31)$$

The solution can be easily seen to be of the form $A\cdot R^{(p-2)/2}\cdot I_{\frac{1}{2}(p-2)}(R)$

or

$$\phi(x)=A\cdot(2\sqrt{x})^{(p-2)/2}\cdot I_{\frac{1}{2}(p-2)}(2\sqrt{x}) \qquad \ldots \quad (3.32)$$

Therefore if A is so chosen that

$$\frac{e^{1/t}}{t^{p/2}} = A \cdot \int_0^\infty (2\sqrt{x})^{(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}(2\sqrt{x}) \cdot e^{-xt} \cdot dx \qquad \ldots \quad (3.33)$$

then

$$\psi_k(t) = \frac{A \cdot \int_0^\infty x_k \cdot (2\sqrt{x})^{(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}(2\sqrt{x}) \cdot e^{-xt} \cdot dx}{e^{1/t}\ t^{-p/2}} \qquad \ldots \quad (3.34)$$

or

$$\psi_k(t) = A \cdot \int_0^\infty x^k \cdot e^{(-1/t-x/t)} \cdot t^{p/2} \cdot (2\sqrt{x})^{(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}(2\sqrt{x}) \cdot dx \qquad \ldots \quad (3.35)$$

Remembering that $\psi_k(t) = \dfrac{\mu_k}{(2st)^k}$ ... (3.36)

We have $\psi_k = A \cdot \int_0^\infty (2stx)^k \cdot e^{(-1/t-x/t)} \cdot t^{p/2}(2\sqrt{x})^{(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}(2\sqrt{x}) \cdot dx$ ... (3.37)

Assuming $(2stx) = r^2$ as a new variable, then

$$\psi_k = A \cdot \int_0^\infty r^{2k} \cdot e^{-(1/t+r^2/2s)}\ t^{p/2} \cdot \{2\sqrt{(r^2/2st)}\}^{(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}\{2\sqrt{(r^2/2st)}\} \cdot \frac{d(r^2)}{2st} \qquad (3.38)$$

Remerbering that

$$t = \frac{2s}{\Delta^2}, \quad \frac{r^2}{2st} = \frac{r^2\Delta^2}{4s^2} \quad \text{or} \quad 2\sqrt{\frac{r^2}{2st}} = \frac{r\Delta}{s} \qquad \ldots \quad (3.39)$$

we have $\mu_k = A \cdot \int_0^\infty \dfrac{r^{2k}}{s} \cdot e^{-(\Delta^2+r^2)/2s} \cdot 2^{(p-1)/2} \cdot \dfrac{r^{p/2}dr}{\Delta^{(p-2)/2}} \cdot I_{\frac{1}{2}(p-2)}\left(\dfrac{r\Delta}{s}\right)$ ... (3.41)

$$= 2^{(p-1)/2} \cdot A \cdot \int_0^\infty \frac{r^{2k} \cdot e^{-(\Delta^2+r^2)2s}}{s \cdot \Delta^{(p-2)/2}} \cdot r^{p/2}dr \cdot I_{\frac{1}{2}(p-2)}\left(\frac{r\Delta}{s}\right) \qquad \ldots \quad (3.42)$$

Now putting $s = 1/n$, we have

$$\mu_k = 2^{(p-1)/2} \cdot A \cdot \int_0^\infty \frac{r^{2k} \cdot n}{\Delta^{(p-2)/2}} \cdot e^{-\frac{1}{2}n(\Delta^2+r^2)} \cdot r^{p/2} \cdot I_{\frac{1}{2}(p-2)}(nr\Delta) \cdot dr \quad \ldots \quad (3.43)$$

Thus the μ's can be regarded as the moment functions for the distribution

$$dF = \text{const.}\ n \cdot \frac{e^{-\frac{1}{2}n(\Delta^2+r^2)}}{\Delta^{(p-2/2)}} \cdot r^{p/2} \cdot I_{\frac{1}{2}(p-2)}(nr\Delta) \cdot dr \qquad \ldots \quad (3.44)$$

which agrees with R. C. Bose's formula.[5]

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[5] R. C. Bose : "On the Exact Distribution and Moment co-efficients of the D²-Statistic". *Sankhyā*, 2(2), 1936, p. 149, equation (6.4).

SPHERICAL FORM OF THE FUNDAMENTAL DIFFERENTIAL EQUATION CONNECTED WITH THE MULTIVARIATE NORMAL POPULATION.

4. I shall now discuss the differential equation

$$\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}+\ldots\ldots+\frac{\partial^2 F}{\partial R_p^2}=2\cdot\frac{\partial F}{\partial t} \qquad \ldots \quad (4.1)$$

which may be regarded as the fundamental equation for the distribution function of the multivariate normal population in p-variates.

If F is a function of (R, t) where

$$R^2=x_1^2+x_2^2+x_3^2+\ldots+x_p^2 \qquad \ldots \quad (4.2)$$

then F satisfies the differential equation

$$\frac{\partial^2 F}{\partial R^2}+\frac{p-1}{R}\quad\frac{\partial F}{\partial R}=2\cdot\frac{\partial F}{\partial t} \qquad \ldots \quad (4.3)$$

THE DIFFERENTIAL EQUATION IN PARABOLIC FORM.

5. When F is a function of the form

$$F(x_1^2+x_2^2+\ldots+x_p^2,\ a_1x_1+\ldots+a_px_p,\ t)$$

by suitably choosing one axis say y_1, perpendicular to the hyperplane

$$a_1x_1+a_2x_2+\ldots\ a_px_p=0 \qquad \ldots \quad (5.1)$$

and similar orthogonal axis-system as before, F takes the form

$$F(R, y_1, t) \text{ or } F(y_1, y_1^2+\ldots+y_p^2, t)$$

The equation (4.1) can be transformed to the form

$$\frac{\partial^2 F}{\partial y_1^2}+\frac{1}{r^{p-2}}\cdot\frac{\partial}{\partial r}\left(r^{p-2}\cdot\frac{\partial F}{\partial r}\right)=2\cdot\frac{\partial F}{\partial t} \qquad \ldots \quad (5.2)$$

where

$$r^2=y_2^2+y_3^2+\ldots+y_p^2 \qquad \ldots \quad (5.2)$$

Further decomposition can be carried on in the same way.

If F is of the form $F(x_1, x_2, R, t)$; the differential equation can be written in the form

$$\frac{\partial^2 F}{\partial x_1^2}+\frac{\partial^2 F}{\partial x_2^2}+\frac{1}{\omega^{p-3}}\cdot\frac{\partial}{\partial\omega}\left(\omega^{p-3}.\frac{\partial F}{\partial\omega}\right)=2\cdot\frac{\partial F}{\partial t} \qquad \ldots \quad (5.4)$$

where

$$\omega^2=x_2^2+x_3^2+\ldots+x_p^2, \text{ and so on.} \qquad \ldots \quad (5.5)$$

THE DIFFERENTIAL EQUATION IN THE PRODUCT FROM.

6. If $F(x_1, x_2, \ldots x_r, t)$ can be written as the product of two functions

$$\phi_1(x_1, x_2, \ldots x_r, t) \cdot \phi_2 (x_{r+1}, x_{r+2}, \ldots x_p, t) \quad \ldots \quad (6.1)$$

then the fundamental differential equation (4.1) breaks up at once to the form

$$\frac{1}{\phi_1}\left[\Box_r^2 \phi_1 - 2\cdot \frac{\partial \phi_1}{\partial t}\right] + \frac{1}{\phi_2}\left[\Box_{r+1 \cdots p}\phi_2 - 2\cdot \frac{\partial \phi_2}{\partial t}\right] = 0 \quad \ldots \quad (6.2)$$

so that the functions ϕ_1 and ϕ_2 satisfy respectively equations of the form

$$\left.\begin{aligned} &\Box_{1\cdot 2 \cdots r}^2 \phi_1 - 2\cdot \frac{\partial \phi_1}{\partial t} = F(t, a)\phi_1 \\ \text{and} \quad &\Box_{r+1 \cdots p}^2 \phi_2 - 2\cdot \frac{\partial \phi_2}{\partial t} = -F(t, a)\phi_2 \end{aligned}\right\} \quad \ldots \quad (6.3)$$

Now these equations are transformable to the forms

$$\left.\begin{aligned} &\Box_r^2 \phi_1 \cdot e^{-\frac{1}{2}\int F(t, a)\, dt} = 2\frac{d}{dt}\left[\phi_1 \cdot e^{-\frac{1}{2}\int F(t, a)\, dt}\right] \\ &\Box_{r+1 \cdots p}^2 \varphi_2 \cdot e^{\frac{1}{2}\int F(t, a)\, td} = 2\,\frac{d}{dt}\left[\phi_2 \cdot e^{\frac{1}{2}\int F(t, a)\, dt}\right] \end{aligned}\right\} \quad \ldots \quad (6.4)$$

Therefore if L_1 and L_2 are solutions of the equations (6.4), their product $L_1 \cdot L_2$ is a solution of the original equation (4.1). It is easily seen that

$$\frac{e^{-x^2/2t}}{t^{1/2}}$$

satisfies the differential equation

$$\frac{\partial^2 F}{\partial x_1^2} = 2\cdot \frac{\partial F}{\partial t} \quad \ldots \quad (6.5)$$

Hence

$$\frac{e^{-\Sigma_r (x_r^2/2t)}}{t^{p/2}}$$

is a solution of the equation in p-variables *i.e.* of the equation

$$\Box_p^2 F = 2\cdot \frac{\partial F}{\partial t} \quad \ldots \quad (6.6)$$

FURTHER REDUCTION OF THE PARABOLIC FORM TO POLAR FORM

7. The equation (5.2) can be written in the form

$$\frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial r^2} + \frac{p-2}{r} \cdot \frac{\partial F}{\partial r} = 2 \cdot \frac{\partial F}{\partial t} \qquad \ldots \quad (7.1)$$

Since

$$R^2 = r^2 + x_1^2 \qquad \ldots \quad (7.11)$$

we can introduce in place of r and x_1, variables R and θ defined by the equations

$$r = R \sin\theta \quad \text{and} \quad x_1 = R\cos\theta \qquad \ldots \quad (7.2)$$

Then

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial p^2} = \frac{\partial^2 F}{\partial R^2} + \frac{1}{R} \cdot \frac{\partial F}{\partial R} + \frac{1}{R^2} \cdot \frac{\partial^2 F}{\partial \theta^2} \qquad \ldots \quad (7.31)$$

and

$$\frac{\partial F}{\partial r} = \sin\theta \cdot \frac{\partial F}{\partial R} + \frac{\cos\theta}{R} \cdot \frac{\partial F}{\partial \theta} \qquad \ldots \quad (7.32)$$

The transformed equation in R, θ becomes

$$\frac{\partial^2 F}{\partial R^2} + \frac{1}{R} \cdot \frac{\partial F}{\partial R} + \frac{1}{R^2} \cdot \frac{\partial^2 F}{\partial \theta^2} + \frac{(p-2)}{R\sin\theta} \cdot \left[\sin\theta \cdot \frac{\partial F}{\partial R} + \frac{\cos\theta}{R} \cdot \frac{\partial F}{\partial R}\right] = \cdot 2\, \frac{\partial F}{\partial t} \ldots \quad (7.33)$$

or

$$\frac{\partial^2 F}{\partial R^2} + \frac{(p-1)}{R} \cdot \frac{\partial F}{\partial R} + \frac{(p-2)}{R} \cdot \cot\theta \cdot \frac{\partial F}{\partial \theta} + \frac{1}{R^2} \cdot \frac{\partial^2 F}{\partial \theta^2} = 2 \cdot \frac{\partial F}{\partial t} \qquad \ldots \quad (7.34)$$

Since

$$x_1^2 + x_2^2 + \omega^2 = R^2 \qquad \ldots \quad (7.41)$$

the equation

$$\frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial \omega^2} + \frac{(p-3)}{\omega} \cdot \frac{\partial F}{\partial \omega} = 2 \cdot \frac{\partial F}{\partial t} \qquad \ldots \quad (7.42)$$

can be transformed by the substitutions

$$\omega = R\cos\theta, \quad x_1 = R\sin\theta\cos\phi, \quad x_2 = R\sin\theta\sin\phi \qquad \ldots \quad (7.51)$$

to

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \cdot \frac{\partial F}{\partial R}\right) + \frac{1}{R^2\sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \cdot \frac{\partial F}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta} \cdot \frac{\partial^2 F}{\partial \phi^2}$$

$$+ \frac{(p-3)}{R\sin\theta}\left[\frac{\partial F}{\partial R} \cdot \cos\theta - \frac{\sin\theta}{R} \cdot \frac{\partial F}{\partial R}\right] = 2\,\frac{\partial F}{\partial t} \qquad \ldots \quad (7.52)$$

Equation (5.2) could be investigated by assuming a solution of the form

$$F = f((R, t) \cdot P(\theta) \qquad \dots \quad (7.61)$$

so that the equation breaks up in the form

$$\frac{\partial^2 f}{\partial R^2} + \frac{(p-1)}{R} \cdot \frac{\partial f}{\partial R} - \frac{\lambda f}{R^2} = 2\frac{\partial f}{\partial t} \text{ and } \frac{\partial^2 P}{\partial \theta^2} + (p-2) \cdot \cot \theta \cdot \frac{\partial P}{\partial \theta} + \lambda P = 0 \quad \dots \quad (7.62)$$

The equation (5.4) could be similarly investigated by assuming a solution of the form

$$F = f(R, t) \cdot P(\theta) \cdot Q(\phi) \qquad \dots \quad (7.7)$$

so that the equation (7.42) breaks up into the three equations

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial R^2} + \frac{(p-1)}{R} \cdot \frac{\partial f}{\partial R} - \frac{\lambda}{R^2} \cdot f &= 2 \cdot \frac{\partial f}{\partial t} \\ \frac{\partial^2 Q}{\partial \phi^2} + n^2 \cdot Q &= 0 \\ \frac{1}{\sin\theta} \cdot \frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial Q}{\partial\theta}\right) + \frac{n^2}{\sin^2\theta} \cdot \frac{\partial Q}{\partial\theta} - (p-3) \cdot \tan\theta \cdot \frac{\partial Q}{\partial\theta} + \lambda Q &= 0 \end{aligned}\right\} \quad \dots \quad (7.8)$$

SOLUTIONS OF THE DIFFERENTIAL EQUATICN IN DIFFERENT CASES.

8. We shall first consider the equation of the type (4.1). The existence of a simple solution of the type

$$\frac{e^{-\Sigma_r(x_r^2/2t)}}{(2\pi t)^{p/2}} = F_1 \qquad \dots \quad (8.11)$$

is verified by direct substitution.

We shall have occasion to make use of the theorem of Appell and Brill, which has been proved in the case of p-variables earlier[6], that if $H(x_1, x_2, \dots x\ , t)$ is a solution of the differential equation (4.1), then

$$H'(x_1, x_2, \dots \cdot \dots x_p, t) = \frac{e^{-\Sigma_r(x_r^2/2t)}}{(2\pi)^{p/2}} \cdot H\left(\frac{x_1}{t}, \frac{x_2}{t}, \dots \frac{x_p}{t}, -\frac{1}{t}\right) \dots \quad (8.12)$$

is also a solution.

Therefore in the case of a spherically symmetric solution of the type

$$H(R^2, t) = H(x_1, x_2, \dots x_p, t) \qquad \dots \quad (8.13)$$

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[6] *Sankhyā*, 2(4), 1936, p. 393, equation (11.1).

$$H'(x_1, x_2, \ldots, x_p, t) = \frac{e^{-\Sigma_r(x_r^2/2t)}}{(2\pi\ t)^{p/2}} \cdot H\left(\frac{R^2}{t^2}, -\frac{1}{t} \right) \qquad \ldots \quad (8.14)$$

is also a solution.

We therefore transform the equation

$$\frac{\partial^2 F}{\partial R^2} + \frac{p-1}{R} \cdot \frac{\partial F}{\partial R} = 2 \cdot \frac{\partial F}{\partial t} \qquad \ldots \quad (4.3)$$

by the substitution $R^2 = y$, and the equation becomes

$$2y \cdot \frac{\partial^2 F}{\partial y^2} + p \cdot \frac{\partial F}{\partial y} = \frac{\partial F}{\partial t} \qquad \ldots \quad (8.16)$$

Further putting $2t = T$, it becomes ... (8.17)

$$y \cdot \frac{\partial^2 F}{\partial y^2} + \frac{p}{2} \cdot \frac{\partial F}{\partial y} = \frac{\partial F}{\partial T} \qquad \ldots \quad (8.18)$$

First consider the equation (4.3). If we assume

$$F = e^{\lambda t} \cdot \psi(R) \qquad \ldots \quad (8.21)$$

$\psi(R)$ satisfies the differential equation

$$\frac{\partial^2 \psi}{\partial R^2} + \frac{p-1}{R} \frac{\partial \psi}{\partial R} = 2\lambda\psi \qquad \ldots \quad (8.22)$$

whose solution can be written in either of the forms

$$\frac{e^{\lambda t}}{R^{(p-2)/2}} \cdot I_{\frac{1}{2}(p-2)}\, (R\sqrt{2\lambda}) \qquad \ldots \quad (8.23)$$

$$\frac{e^{-\lambda t}}{R^{(p-2)/2}} \cdot J_{\frac{1}{2}(p-2)}(R\sqrt{2\lambda}) \qquad \ldots \quad (8.24)$$

Therefore, from (8.13) and (8.14) we get two further types of solutions. Carrying out the transformation suggested it is seen that

$$\frac{e^{-\Sigma(x_r^2/2t)}}{(2\pi t)^{p/2}} \cdot \frac{e^{-\lambda/t}}{R^{(p-2)/2}} \cdot t^{(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}\{\sqrt{(2\lambda)} \cdot R/t\} \qquad \ldots \quad (8.25)$$

and

$$\frac{e^{-\Sigma(x_r^2/2t)}}{(2\pi t)^{p/2}} \cdot \frac{e^{\lambda/t}}{R^{(p-2)/2}} \cdot t^{(p-2)/2} \cdot J_{\frac{1}{2}(p-2)}\, \{\sqrt{(2\lambda)} \cdot R/t\} \qquad \ldots \quad (8.26)$$

are also solutions of the equation (4.3).

Hence writing $\lambda = \xi^2/2$, we have as solutions

$$\frac{e^{-(\xi^2+R^2)/2t}}{t\cdot(2\pi)^{p/2}\cdot R^{(p-2)/2}} I_{\frac{1}{2}(p-2)}(R\xi/t) \qquad \dots \quad (8.27)$$

and

$$\frac{e^{-(R^2-\xi^2)/2t}}{t\cdot(2\pi)^{p/2}\cdot R^{(p-2)/2}} J_{\frac{1}{2}(p-2)}(R\xi/t) \qquad \dots \quad (8.28)$$

We shall now seek polynomial types of solution. Stating from the formula (4.3) and putting

$$y = R^2,\ T = 2t \qquad \dots \quad (8.31)$$

we get

$$y\cdot\frac{\partial^2 F}{\partial y^2} + \frac{p}{2}\cdot\frac{\partial F}{\partial y} = \frac{\partial F}{\partial T} \qquad \dots \quad (8.32)$$

Let us put

$$F = T^k\cdot\phi(y/T) \qquad \dots \quad (8.33)$$

Since

$$\frac{\partial F}{\partial y} = T^{k-1}\cdot\phi',\ \frac{\partial^2 F}{\partial y^2} = T^{k-2}\cdot\phi'',\ \frac{\partial^2\phi}{\partial T^2} = k\cdot T^{k-1}\cdot\phi - y\cdot T^{k-2}\cdot\phi \qquad \dots \quad (8.34)$$

the equation (8.32) becomes

$$y\cdot T^{k-2}\phi'' + (p/2)\cdot T^{k-1}\cdot\phi' = kT^{k-1}\cdot\phi - y\cdot T^{k-2}\cdot\phi' \qquad \dots \quad (8.35)$$

or dividing by T^{k-1} and putting $y/T = s$ ... (8.36)

$$s\cdot\phi_k'' + \{(p/2)+s\}\cdot\phi_k' - k\cdot\phi_k' = 0 \qquad \dots \quad (8.37)$$

This equation has been already discussed in a previous paper.[7]

It is easily seen that with integer values of k, it gives polynomial solutions. These solutions are denoted by $\mu_k(y, T)$ here. They are identical with the moment-functions discussed at some length in the earlier paper.[8] The solutions can be written in the form

$$T^k\cdot\mu_k(y/T) \qquad \dots \quad (8.41)$$

The application now of (8.12), allows us to deduce solutions of the type

$$\frac{e^{-R^2/2t}}{(t)^{p/2}}\cdot\frac{1}{T^k}\cdot S\,(s),\ \text{where } s = y/T = R^2/2t \qquad \dots \quad (8.42)$$

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[7] *Sankhyā*, 2(4), 1936, p. 390, equation (8.5).

[8] *Sankhyā*, 2(4), 1936 pp. 387—390.

and
$$s \cdot S''_k + \{(p/2)-s\} \cdot S'_k + k \cdot S_k = 0 \qquad \ldots \quad (8.43)$$

We therefore see that

$$\frac{e^{-R^2/2t}}{(2\pi t)^{p/2}} \; \frac{1}{T^{2k}} \cdot S_k(y, T) \qquad \ldots \quad (8.44)$$

is also a solution where $S_k(y, T)$ is a homogeneous polynomial in (y, T).

We shall now seek relations between solutions of type (4.3) and (5.3) with type (8.44). We note that since (8.44) is a solution of the fundamental equation (4.3), writing

$$R^2 = y, \; 2t = T \qquad \ldots \quad (8.45)$$

we see that

$$L = \frac{e^{-y/T}}{T^{p/2+k}} \cdot S_k(y/T) \text{ satisfies the equation (8.32)} \qquad \ldots \quad (8.46)$$

that is,
$$y \cdot \frac{\partial^2 L}{\partial y^2} + \frac{p}{2} \cdot \frac{\partial L}{\partial y} = \frac{\partial L}{\partial T} \qquad \ldots \quad (8.47)$$

This result will be useful later on.

Consider the fundamental equation (8.32) in (y T).

$$y \cdot \frac{\partial^2 \psi}{\partial y^2} + \frac{p}{2} \cdot \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial T} \qquad \ldots \quad (8.32)$$

If we write
$$\psi = e^{\lambda T} \cdot F(y) \qquad \ldots \quad (8.51)$$

we can see that

$$y \cdot \frac{\partial^2 F}{\partial y^2} + \frac{p}{2} \cdot \frac{\partial F}{\partial y} = \lambda \cdot F \qquad \ldots \quad (8.52)$$

The form of the equation shows that

$$\psi = e^{\lambda T} \cdot F(\lambda y) \qquad \ldots \quad (8.53)$$

is a solution of (8.52) where $F(y)$ satisfies the equation

$$\frac{\partial^2 F}{\partial y^2} + \frac{p}{2y} \cdot \frac{\partial F}{\partial y} = \frac{F}{y} \qquad \ldots \quad (8.54)$$

Appell's theorem applied to this form gives us a solution of the form

$$t^{-p/2} \cdot e^{-R^2/2t} \cdot e^{-\lambda/T} \cdot F(\lambda y/T) \qquad \ldots \quad (8.55)$$

or
$$t^{-p/2} \cdot e^{-R^2/t^2} \cdot F(\lambda R^2/T) \qquad \ldots \quad (8.56)$$

Remembering that
$$T = 2t, \quad \text{and putting} \quad \lambda = \Delta^2 \qquad \ldots \quad (8.57)$$

the solution reduces to

$$t^{-p/2} \cdot e^{-(R^2+\Delta^2)/2t} \cdot F \cdot (\Delta^2 R^2/4t) \qquad \ldots \quad (8.54)$$

Remembering again that $T = 2t$, we see that the solution of (8.54) is of the form

$$t^{-p/2} \cdot e^{-R^2/2t} \cdot e^{-\Delta^2/2t} \cdot \psi(\Delta R/t) \qquad \ldots \quad (8.61)$$

Now a solution of the above type gives rise to a distribution

$$dF = H \cdot dx_1 \cdot dx_2 \ldots dx_p \qquad \ldots \quad (8.62)$$

where H is a solution of the equation. Therefore

$$dF = A \cdot e^{-(R^2+\Delta^2)/2t} \cdot t^{-p/2} \cdot \psi(\Delta R/t) \cdot R^{(p-2)/2} \qquad \ldots \quad (8.63)$$

If we compare the result with that in the earlier paper, it is easily seen that the function

$$\psi(\Delta R/t) = A \cdot (n\lambda l)^{-(p-2)/2} \cdot I_{\frac{1}{2}(p-2)} \cdot (nl\lambda) \qquad \ldots \quad (8.64)$$

using the notation of R. C. Bose's paper[9].

If we remember that $n = 1/t$, $l = R$, $\lambda = \Delta$ the identity of the solution can also be easily seen.

By transforming the equation

$$Z \cdot \frac{\partial^2 F}{\partial Z} + \frac{p}{2} \cdot \frac{\partial F}{\partial Z} = F \qquad \ldots \quad (8.72)$$

by the substitution $Z = x^2/4$, we have

$$\frac{\partial^2 F}{\partial x^2} + \frac{(p-1)}{x} \cdot \frac{\partial F}{\partial x} = F \qquad \ldots \quad (8.73)$$

Since I_n/x satisfies the equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{(2n+1)}{x} \cdot \frac{\partial F}{\partial x} = F \qquad \ldots \quad (8.74)$$

the solution of the above is easily seen to be

$$A \cdot x^{-(p-2)/2} \cdot I_{\frac{1}{2}(p-2)}(x) \qquad \ldots \quad (8.75)$$

ON THE SERIES SOLUTION OF THE DIFFERENTIAL EQUATION.

9. I have assumed in an earlier paper the possibility of expansion of solutions or the equation (4.1) as series of the type[10]

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[9] Raj Chandra Bose: "On the Exact Distribution and Moment-coefficients of the D²-statistic" *Sankhyā*, 2(2), 1936, p. 148, equation (6.4). [10] *Sankhyā*, 2(4), 1936, p. 396, equation (12.7).

$$\Sigma_k\, A_k \cdot \frac{e^{-R^2/2t}}{t^{p/2}} \cdot \frac{1}{t^k} \cdot S_k(R^2/2t)$$

where Σ_k denotes a summation over all values of k from 1 to ∞, and S_k denotes a Sonine polynomial of the k-th order

The justification for the statement will be given here, and *I* shall, by actually developing the distribution function, arrive at the same result as before. A distribution function of the type

$$t^{-p/2} \cdot e^{-(\xi^2+R^2)/2t} \cdot F(\xi^2R^2/4T^2)$$

where $R^2 = x_1^2 + x_2^2 + \ldots + x_p^2$, necessarily implies that

$$\int t^{-p/2} \cdot e^{-(\xi^2+R^2)/2t} \cdot F(\xi^2R^2/4t^2) \cdot R^{p-1} \cdot dR = 1 \qquad \ldots \quad (9.12)$$

Assume now that $$R^2/2t = Z, \quad \text{and} \quad \xi^2/2t = a \qquad \ldots \quad (9.13)$$

The integral then takes the form

$$2^{(p-2)/2} \cdot \int_0^\infty t^{-p/2} \cdot e^{-(a+z)} \cdot F(az) \cdot z^{(p-2)/2} \cdot t^{p/2} \cdot dZ = 1 \qquad \ldots \quad (9.14)$$

or $$2^{(p-2)/2} \cdot e^{-a} \int_0^\infty e^{-z} \cdot F(az) \cdot z^{(p-2)/2} \cdot dz = 1 \qquad \ldots \quad (9.15)$$

Therefore so long as a remains finite

$$\int_0^\infty e^{-z} \cdot z^{(p-2)/2} \cdot F(az) \,.\, dz = 2^{-(p-2)/2} \cdot e^a = \text{finite} \qquad \ldots \quad (9.16)$$

Therefore a constant c can be found such that

$$\int_0^\infty e^{-z} \cdot z^{(p-2)/2} \cdot [F(az)-c] \cdot dz = 0 \qquad \ldots \quad (9.17)$$

so that if we consider the function $$F(az)-c \equiv F_1(z) \qquad \ldots \quad (9.18)$$

we have $$\int_0^\infty e^{-z} \cdot z^{(p-2)/2} \cdot F_1(z) \cdot dz = 0 \qquad \ldots \quad (9.19)$$

Now the Sonine polynomials $S_k(z)$ satisfy the differential equation

$$z \cdot \frac{d^2S_k}{dz^2} + \left(\frac{p}{2} - z\right) \cdot \frac{dS_k}{dz} + k \cdot S_k = 0 \qquad \ldots \quad (9.21)$$

or written in the self-adjoint form

$$\frac{d}{dz}\cdot\left(z^{p/2}\cdot e^{-z}\cdot\frac{dS_k}{dz}\cdot\right) + k\cdot S_k\cdot e^{-z}\cdot z^{(p-2)/2} = 0 \quad \ldots \quad (9.22)$$

We see that $e^{-z}\cdot z^{(p-2)/2}$ is the density function for the orthogonal set of polynomials. Hence follows the possibility of expansion of a function $F_1(z)$ which satisfies the equation

$$\int_0^\infty e^{-z}\cdot z^{(p-2)/2}\cdot F_1(z)\cdot dz = 0 \quad \ldots \quad (9.23)$$

Assuming therefore that $F(az) = \Sigma_k\, A_k\cdot S_k(z)$ ... (9.24)

the co-efficients A_k will be (because of the orthogonality of the polynomial functions S_k mentioned earlier) found to be given by

$$A_k = \frac{\int_0^\infty F(az)\cdot S_k(z)\cdot e^{-z}\cdot z^{(p-2)/2}\cdot dz}{\int_0^\infty e^{-z}\cdot z^{(p-2)/2}\cdot S_k(z)\cdot dz}$$

Let us suppose that the functions are normalized such that

$$\int_0^\infty e^{-k}\cdot z^{(p-2)/2}\cdot S_k^2(z)\cdot dz = 1 \quad \ldots \quad (9.26)$$

Then

$$A_k = \int_0^\infty F(az)\cdot S_k(z)\cdot e^{-z}\cdot z^{(p-2)/2}\cdot dz \quad \ldots \quad (9.27)$$

To calculate A_k we at first write $az = x;$ then ... (9.28)

$$A_k = \int_0^\infty F(x)\cdot S_k(z/a)\cdot a^{-p/2}\cdot e^{-x/a}\cdot x^{(p-2)/2}\cdot dx \quad \ldots \quad (9.31)$$

As noted in *Sankhyā*, 2(4), 1836, pp. 394, $L = a^{-(p+2k)/2}\cdot e^{-x/a}\cdot S_k(x/a)$ ... (9.32)
satisfies the differential equation

$$x\cdot\frac{\partial^2 L}{\partial x^2} + \frac{p}{2}\cdot\frac{\partial L}{\partial x} = \frac{\partial L}{\partial a} \quad \ldots \quad (9.33)$$

$$A_k/a^k = \int_0^\infty F(x)\cdot x^{(p-2)/2}\cdot L\cdot dx \quad \ldots \quad (9.41)$$

or

$$\frac{\partial}{\partial a}\left(\frac{A_k}{a_k}\right) = \int_0^\infty x^{(p-2)/2} \cdot F(x) \cdot \frac{\partial L}{\partial a} \cdot dx \qquad \ldots \quad (9.42)$$

$$= \int_0^\infty x^{(p-2)/2} \cdot F(x) \cdot \left[x \cdot \frac{\partial^2 L}{\partial x^2} + \frac{p}{2} \cdot \frac{\partial L}{\partial x} \right] \cdot dx \qquad \ldots \quad (9.43)$$

Now by successive integration by parts, because the quantities outside the sign fo integration vanish at both limits, it easily follows that

$$\frac{\partial}{\partial a}\left(\frac{A_k}{a_k}\right) = \int_0^\infty L \cdot \left[\frac{\partial^2}{\partial x^2} \cdot (x^{p/2} \cdot F) - \frac{p}{2} \cdot \frac{\partial}{\partial x} (x^{\frac{1}{2}p-1} \cdot F) \right] \cdot dx \ldots \qquad (9.44)$$

$$= \int_0^\infty L \cdot x^{p/2} \cdot \left[\frac{\partial^2 F}{\partial x^2} + \frac{p}{2x} \cdot \frac{\partial F}{\partial x} \right] \cdot dx \qquad \ldots \quad (9.45)$$

As

$$\frac{\partial^2 F}{\partial x^2} + \frac{p}{2x} \cdot \frac{\partial F}{\partial x} = \frac{F(2x)}{x} \qquad \ldots \quad (9.46)$$

we have now

$$\frac{\partial}{\partial \alpha}\left(\frac{A_k}{a_k}\right) = \int_0^\infty L \cdot x^{(p-2)/2} \cdot F_{(x)} \cdot dx = \int_0^\infty a^{-(p+2k)/2} \cdot e^{-x/a} \cdot S_k(x/a) \cdot x^{(p-2)/2} F_{(x)} \cdot dx \qquad \ldots \quad (9.51)$$

Writing $x = az$ as before, we have

$$\frac{\partial}{\partial a}\left(\frac{A_k}{a_k}\right) \frac{1}{a^k} = \frac{1}{a^k} \int_0^\infty e^{-z} \cdot S_k(z) \cdot z^{(p-2)/2} \cdot F(az) \cdot dx \qquad \ldots \quad (9.52)$$

Therefore we have

$$\frac{\partial}{\partial a} \cdot \left(\frac{A_k}{a^k}\right) = \frac{A_k}{a^k} \qquad \ldots \quad (9.61)$$

or

$$A_k = C \cdot a^k \cdot e^a \qquad \ldots \quad (9\ 62)$$

To detrmine the constant C in the expression for A_k we observe that

$$\int_0^\infty F \cdot (az) \cdot e^{-z} \cdot z^{(p-2)/2} \cdot S_k(z) \cdot dz = C \cdot a^k \cdot e^a \qquad \ldots \quad (9.63)$$

But when a is very small $C \cdot a^k \cdot e^a = C \cdot a^k \cdot (1+a+a^2/2+\ldots) = C \cdot a^k \ldots$ (9.64)

Therefore

$$C \cdot a^k = \int_0^\infty B_k \cdot a^k \cdot z^k \cdot e^{-z} \cdot z^{(p-2)/2} \cdot S_k(z) \cdot dz \qquad \ldots \quad (9.65)$$

where B_k is the coefficient of z^k in the expansion of $F(z)$ as a power series, as terms of lower order in $F(a\,z)$ below z^k, will contribute zero values because of the orthogonal properties of $s_k(z)$'s

Now as $$\int_0^\infty z^k \cdot e^{-z} \cdot z^{(p-2)/2} \cdot S_k(z) \cdot dz = \frac{1}{\gamma_k} \int_0^\infty e^{-z} \cdot z^{(p-2)/2} \cdot S_k^2 \cdot (z) \cdot dz = \frac{1}{\gamma_k} \ldots \quad (9.66)$$

where γ_k is the coefficient of the highest term in the series of normalised Sonnine polynomials, we have

$$C \cdot a^k = \frac{B_k \cdot a^k}{r_k}, \text{ or } \quad C = \frac{B^k}{\gamma^k} \quad \ldots \quad (9.71)$$

To calculate therefore the coefficients of B_k and γ_k we consider the differential equation which $F(z)$ satisfies, that is,

$$z \cdot \frac{d^2F}{\partial z^2} + \frac{p}{2} \cdot \frac{\partial F}{\partial z} = F(z) \quad \ldots \quad (9.81)$$

Assuming a series solution $$F \equiv \Sigma B_k \cdot z^k \quad \ldots \quad (9.82)$$

is easily seen that $$k \cdot (k-1) \cdot B_k + \tfrac{1}{2}p \cdot k \cdot B_k = B_{k-1} \quad \ldots \quad (9.83)$$

or $$k \cdot (k+\tfrac{1}{2}p-1) \cdot B_k = B_{k-1} \quad \ldots \quad (9.84)$$

or $$B_k = \frac{B\Gamma\ (p/2)}{\Gamma(k+1) \cdot \Gamma(k+\frac{1}{2}p)} \quad \ldots \quad (9.85)$$

Therefore from (9.24), (9.62), (9.66) and (9.85) we have

$$F(az) = B_0\, \Sigma \frac{\Gamma(p/2)}{\Gamma(k+1)\, \Gamma(k+\frac{1}{2}p)} \cdot a^k\, e^a \cdot S_k'\,(z) \quad \ldots \quad (9.86)$$

where $$S'(z) = S_k(z)/\gamma_k \quad \ldots \quad (9.87)$$

Putting now $$a = \xi^2/2t, \quad \text{and} \quad z = R^2/2t \quad \ldots \quad (9.88)$$

we see that

$$F\left(\frac{\xi^2 R^2}{4t^2} \right) = F(0)\, \Sigma \frac{\Gamma(p/2)}{\Gamma(k+1)\, .\, \Gamma(k+\frac{1}{2}p)} \cdot e^{\,\xi^2/2t} \cdot (\xi^2/2t)^k \cdot S_k(R^2/2t) \ldots \quad (9.89)$$

or $$F(x) = F(0)\Sigma \frac{\Gamma(p/2)}{\Gamma(k+1) \cdot \Gamma(k+\frac{1}{2}p)} \cdot e^{\xi^2/2t} \cdot (\xi^2/2t)^k \cdot S_k \cdot \left(\frac{\xi^2}{2t} \cdot x \right) \ldots \quad (9.91)$$

Therefore the type of solution is of the form

$$\frac{e^{-(\xi^2+R^2/2t}}{t^{p/2}} \quad F(\xi^2R^2/4t^2)$$

$$= F(0)\Sigma \frac{\Gamma(p/2)}{\Gamma(k+1)\cdot\Gamma(k+\frac{1}{2}p)} \cdot \frac{e^{-R^2/2t}}{t^{p/2}} \cdot \frac{\xi^{2k}}{(2t)^k} \cdot S_k \, . \, (R^2/2t) \qquad \ldots \quad (9.92)$$

which agrees with the previous result[11], if we remember the expression

$$\Delta^k = \xi^{2k} = \mu_k - k(k + \tfrac{1}{2}p - 1)\cdot\mu_{k-1}(2t) + \ldots \qquad \ldots \quad (9.93)$$

It follows from the above analysis that the general form of the distribution function in (R, t) can be assumed to be of the form

$$\Sigma \frac{e^{-R^2/2t}}{(2\pi t)^{p/2}} \left\{ A_0 + \frac{A_1}{t^2} \cdot S_1(R^2/2t) + \frac{A_2}{t^4} S_2(R^2/2t) + \ldots \right\} \qquad \ldots \quad (9.94)$$

where $S_1, S_2, \ldots S_k$ are what can be called Sonine polynomials. The coefficients $A_0, A_1, A_2 \ldots$ etc. can be calculated in any particular case, when the moments of the modulus distributions are known. The distribution-function in *Sankhya*[12] may thus be seen to fall in the general class, as I have shown by a direct expansion of the expression, and also by identification of the two results.

It is interesting to stydy the general types of solutions of the equation $\Box_p^2 F = 2\frac{\partial F}{dt}$ for different types of complexity; certain results already arrived at will be published in due course.

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[11] *Sankhyā*, 2(4), pp. 395—396.

[12] *Sankhyā*, 2(4), 1936, pp. 395-396.

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Recent Progress in Nuclear Physics

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PHYSICAL science is at present passing through a remarkable phase of radical and revolutionary changes. Time-honoured concepts are crumbling down under the shock of remarkable discoveries. The familiar ideas of force, mass, and energy have undergone remarkable transformations and the dynamical laws which seemed at one time to provide the ultimate basis for the mechanical explanation of the material universe have now been replaced to a large extent by quantum-mechanical rules and probability-calculations. Controversy now rages round the validity of the very principle of causality without which science would have seemed impossible a few years ago. It is therefore not surprising that the old concept of the atom as the ultimate *indivisible* unit of a chemically simple stuff has been replaced by the modern idea of a complicated structure consisting of a positively charged material core, the nucleus, which itself in ultimate analysis appears to be heterogeneous, enveloped in a cloud of negative electrons, which possess more or less definite amounts of energy and momentum, determined by quantum-mechanical rules.

The Electron and the Periodic Table

I shall endeavour in this brief discourse to present before you a brief history of the various experiments which have compelled the modern scientists to adopt this structure for the chemical atom in place of the familiar and simple concepts of Lucretius or of Dalton. The beginnings of the change in our point of view may be said to have been initiated about 40 years ago by the discovery of the electron in 1895. The experiments which led to its discovery established at the same time its presence as a universal constituent in all kinds of atoms. The intimate connexion between electricity and matter being thus established once for all, the subsequent endeavours of the physicists and the chemists have been directed towards explaining the divergent properties of the chemical substances in terms of electricity and electrical forces.

It will be obviously impossible in this brief compass to give an adequate account of all the results so far achieved in this region where physicists and chemists have worked side by side, and I shall therefore confine myself to a bare mention of the principle facts which have led the way to the modern conceptions about the atom. The ancient atomic theory was formulated mainly on the basis of chemical evidence. The analysis of innumerable substances, which either occur as such in nature or are artificially produced in the laboratory, has established the existence of about 92 simple substances, the so-called elements, whose atoms by combining and re-combining among themselves in various proportions have given rise to all the various substances we see around us. Without losing their individual distinctive features, many of these elements show among themselves remarkable similarities in their chemical properties. These have been intensively studied by the chemists, and the main results can be conveniently represented by arranging all the elements in a series of horizontal and vertical rows, in the so-called periodic table of Mendeljieff.

The remarkable feature of this arrangement is that whereas the atomic weights of elements increase steadily as we go down the series, marked similarities in the chemical properties recur at more or less regular intervals, *i.e.*, as soon as we come to elements which lie in the same vertical column of the rectangular array. The atomic number in the scheme plays as important or rather a more important role than the atomic weight of elements. No explanation of this regularity can obviously be found in the simple Daltonian theory. Nor was there any prospect from the chemical side alone of arriving at the explanation of the mysterious regularity in the chemical behaviour of elements.

The discovery of the electron however inaugurated novel methods of attack for the solution of the puzzle. The presence of electrons carrying negative charges, as constituent of all atoms, as well as the electrical neutrality of the atom as a whole, had brought the interesting question of the electrical structure of the atom to the fore-front. After the discovery of the electron the atom for the physicist could no longer continue to be a simple substance. The problem of its composition, *i.e.*, the distribution of mass and charge inside it, demanded an urgent solution, and speculations were at work regarding its structure even before adequate data were obtained for drawing probable conclusions.

Closely following on the discovery of the electrons, came the discovery of radium and the radioactive elements. Along with the familiar and stable elements which seem to persist unchanged through geological periods of time, the discovery of the phenomenon of radioactivity established the existence of the so-called unstable elements. Though these behave as ordinary elements in all chemical reactions, they are found to disintegrate spontaneously and give birth to new elements of smaller atomic weights, which in turn explode, regenerate fresh elements, and the series of elements of decreasing atomic weights is continued till the whole process comes to a stop with an element of ordinary stability at the end. During this process of successive disintegration the radioactive elements emit swiftly moving charged particles (the so-called α– or β-radiation). They also emit in general penetrating γ-radiations of the type of X-rays. The α-particles were early recognized as the nuclei of helium atoms which carry two units of positive charge whereas the β-rays were found to be swift electrons which move with velocities approaching the velocity of light. Though it was found impossible to control the phenomena of spontaneous disintegration of the elements, their very existence revealed the composite nature of at least the heavy atoms, and made the hypothesis of a structure for all atoms out of comparatively simpler substances a very probable one.

The swiftly moving positively charged α-particles, emitted during the process of radioactive decay, furnished the physicists with a very convenient weapon for attacking the problem of the constitution of atoms. Lord Rutherford conceived the brilliant idea of sending these swiftly moving charged particles as probing agents inside the atoms, where their mass and enormous kinetic energy would enable them to penetrate far into the mysterious interior, before they would be deviated out of their straight course by the intense Coulombian field of force.

The determination of the distribution of the scattered particles in various azimuths would, he hoped, enable the physicists to obtain a fairly correct picture of the distribution of mass and charge inside the atom. The first experiments in this direction were carried out in Rutherford's Laboratory at Manchester by Geiger and Marsden, and they at once afforded valuable information regarding the probable constitution of the atoms.

The physicists had already arrived at an estimate of the atomic size from various considerations. The kinetic theory indicated the radius to be about 10^{-8}cm. The experiments of Geiger and Marsden now revealed that the mass of the atom must be regarded as concentrated within a sphere of a much smaller radius, say (10^{-12} to 10^{-13}cm.). This central core was also found to be positively charged, and its magnitude was approximately estimated at half the atomic weight of the element. This experiment of Geiger and Marsden enabled the physicists to form a fairly correct idea about atomic exterior. The planetary atomic model suggested by Rutherford, where a positively charged nucleus controls electrons revolving in orbits controlled by Coulombian forces, gained thus a universal acceptance among the scientists and proved a valuable and fruitful hypothesis. Detailed discussion of the subsequent developments will lead us too far away from our main theme. I shall therefore mention in the briefest of terms the principle achievements which we owe to this model. In 1913 Bohr showed that the optical spectra of elements could be explained on the basis of the above model if the electronic constituents of the atoms were supposed to move in definite orbits—determined mainly by electrostatic forces and by subsidiary quantum-laws. The study of the X-ray spectra of elements enabled Moseley at about the same time to settle with certainty the magnitude of the charge of the nucleus and also the number of electrons in outside orbits. The fundamental nature of the atomic number in the Mendeljieff's table received thereby a rational interpretation, and no interpolation, or change of arrangement in the series, could be conceived of at any subsequent period, as the sequence of atoms in the table followed the integral sequences of increase of atomic charges. The idea of the spinning electron gave a rational explanation of the periodicities in the Mendeljieff's table. The electrons were revealed to be grouped in different closed shells round the central core and the chemical properties of the elements could be definitely correlated to the number of the electrons in the outermost incomplete shells. The elements of the same vertical column were found thus to have more or less identical external structure, which explained the similarities observed in their chemical properties.

Structure of the Nucleus

In 1925 came important theoretical developments. The modern wave-mechanics was formulated which enabled the physicists to replace the former tentative calculations by exact mathematical analysis and the different empirical procedures were unified into a single consistent analytical discipline. Though all the questions which have been raised by the study of the physical and chemical behaviour of elements may thus be said to have obtained more or less satisfactory solutions, the problem of the structure of nucleus had however been scarcely touched till the beginning of the present decade.

It is not difficult to find reason for this delay in the development of nuclear physics. The nuclei of elements lie hidden behind a protective cloud of electrons. The intensity of the Coulombian field also increases very rapidly by about 10^{10} times as we approach from the outside to the neighbourhood of the nucleus. The bombardment of atoms by swiftly moving electrons had not produced any fruitful results so far, and in order that positively charged particles could overcome the enormous repulsive force and approach the nucleus within a reasonable distance sufficient to produce significant perturbations in the nuclear regions and produce sensible results, swift particles with tremendous velocities appeared at fisrt sight necessary, which were only available in small amounts from radioactive processes that continue yet to be beyond our control. The smallness of the nuclear size makes also the chances of close collision very very remote so that the percentage yield of any definite result by the bombardment of α-particles is very small indeed. Nevertheless since only the chemical properties of elements appeared to be governed by the nuclear charge, an artificial transmutation of elements could be hoped for, if one were able either to push a charged particle in the interior of the nucleus or to bring a charged particle in its immediate proximity, so that the disturbance thus set up might possibly induce spontaneous transmutation of the atom.

In order to ensure stability against the disruptive influences of the Coulombian forces, the presence inside the atom of attractive forces of unknown origin appeared also necessary. These attractive forces are most probably sensible at distances comparable with the linear dimensions of the nucleus, so that only a careful study of the large deflections of α-particles from single and close encounters with light atoms might be expected to give some information about the nature of these attractive forces. A successful carrying out of the above programme required the development of a special technique for the study of such atomic encounters. The early method of directly counting the scintillations to estimate the large angle scattering of α-particles as followed by Geiger and Marsden was further improved upon by Chadwick and Bieler in the laboratory of Rutherford at Cambridge. A direct measurement of the nuclear charge and an estimate of the nature of the electric forces acting on α-particles in the immediate neighbourhood of light nuclei were rendered possible by the study of scintillations produced by the scattered particle.

The expansion apparatus of Wilson supplied another valuable method of attack. Under suitable circumstances, the tracks of the colliding particles, the nucleus and the α-projectiles, before and after collision, could be directly photographed, and the interchange of momentum and energy between the colliding particles could be directly estimated from the measurement of stereoscopic pictures of such encounters. In case such collisions brought about artificial disintegration, the record of the explosion in the chamber photograph enabled us in many cases to follow the details of the process unequivocally.

The method of directly counting the scattered particles by the scintillations produced on screens has been replaced in recent times by simple, reliable, and automatic methods of counting the swift particles like α-particles or protons with the help of the Geiger Counter and proportionate amplifiers. The rapidity of advance during the last few years has been in large part due to the great improvements in the technical methods of attack.

Nuclear Disintegration; the Neutron

Already however in 1919 Rutherford had obtained by the simple scintillation method the first evidences of artificial disintegration. Nitrogen nucleus bombarded by swiftly moving α-particles appeared to give rise to streams of swiftly moving protons or nuclei of hydrogen, and subsequent work along the same direction had revealed the occurrence of this phenomena of artifical disintegrations during the bombardment of about twelve of the lightest elements.

By studying the phenomena of disintegration of nitrogen in a Wilson Chamber, Blackett was able in 1925 to give a satisfactory account of the details of the process. This nuclear reaction seems to arise out of a capture of the α–particle by the nucleus of nitrogen, whereby an H-particle escapes out of the nucleus and a new atom, an isotope of oxygen, is produced, according to the scheme

$$N_7^{14} + He_2^4 \rightarrow H_1^1 + O_8^{17} .$$

Two capital discoveries in recent years have revolutionized the subject and accelerated the progress of nuclear physics. Bothe in 1930 observed that beryllium when bombarded by α-rays from polonium gave rise to a markedly penetrating radiation which appeared to be of the γ-ray type. In a subsequent examination of this effect by M and Mme Curie Joliot in Paris and Chadwick in Cambridge in 1932, an important part of this radiation was found to consist of a stream of swift, uncharged particles, called neutrons, which have about the same mass as the proton. As it is uncharged the neutron does not directly ionize the gas in its path, but only reveals its presence indirectly by the recoil of the nucleus with which it collides. And since the transfer of momentum is largest when the mass of the colliding nucleus is about the same as the mass of the neutron, the secondary ionization by recoil, as produced in hydrogen or in paraffin-lined ionization-chambers, is much larger than in an ionization-chamber filled with a heavier gas. This peculiarity of behaviour has served to differentiate the neutron radiation from wave radiations of the γ-ray type, which also usually accompany the phenomena of neutron emission. Most of the nuclei of the light atoms also emit neutrons when bombarded by swiftly moving α-particles. This strange type of particles is in itself an important agent for effecting artificial transformation of other nuclei, mainly because it is uncharged and as such is not handicapped by the presence of the intense Coulombian forces. It can thus approach and penetrate into the nuclei of even the heavy atoms, and thereby bring out interesting transformations in novel ways about which I shall speak later on.

Potential Barrier

The perfection of wave-mechanical methods has induced various theoretical workers to apply the new ideas in problems of nuclear physics, especially in investigating the collision of charged particles with the atomic nuclei. The simultaneous presence of a Coulombian field of repulsion and an attractive force of unknown origin at short distances inside the nucleus gives rise to what is technically called the potential barrier. The intense attractive forces inside the barrier serve to keep the charged particles inside the nucleus. Beyond the distance where the height of the barrier rises to a maximum, the attractive forces cease to be sensible, and the field may be ragarded as repelling charged particles according to the law of inverse squares. Any charged particle wanting to penetrate inside the nucleus will have to surmount the barrier, and according to the classical theory, it will have to have a kinetic energy sufficient to come over the top of the barrier.

Based on arguments similar to the above, one estimated (before the advent of the wave-mechanics) that a minimum acceleration-potential of a few million volts would be necessary to produce sufficient acceleration in particles, before they are able to cross the barrier. The estimate however proved erroneous and excessive.

A noteworthy contribution of the theoretical physicists (*recent wave-mechanics)* in this subject has been to predict a small but appreciable probability for penetration of protons across the barrier of light elements, even when they have energies of the order of a few hundred thousand e.volts. The prediction seemed to bring artificial transmutation on a large scale within the range of possibilities, and this theoretical conclusion was tested experimentally by Cockroft and Walton in 1932 who had been able to generate a proton stream of about 100,000 volts, in a vacuum tube, by step-wise acceleration. Their experiments at once met with complete success and the evidence of disintegration of the lithium nucleus subsequent to the capture of a proton was obtained by bright scintillations produced on the screen by the α-particles that were generated by disintegration. This initial success of the Cambridge physicists has been followed up by other workers in Germany, France, and America, and it has become clear that considerable progress in artificial disintegration can be achieved by bombarding elements with swiftly moving protons and other nuclei accelerated suitably by application of high voltages. Investigators of nuclear physics have concentrated their energies on the production of suitable high voltages, and notable success in this direction has been achieved in recent years. Three different methods have been mainly followed in producing the necessary high voltage : firstly, the original method of Cockroft and Walton, which consists in accelerating the particles in the vacuum tubes in stages with suitably insulated transformers; secondly, the method of Lawrence, who has developed a peculiar method of multiple acceleration of ions in a synchronized magnetic and electric alternating field; and thirdly, the electrostatic method of Van de Graff. Considerable progress has been achieved by the application of all the three methods and a lot of interesting results have ben obtained by the bombardment of atomic nuclei, by suitably accelerated projectiles. I have already mentioned before the production of neutrons from the light elements by the bombardment of α-particles. Results of great significance were obtained as soon as these particles were used by workers in nuclear physics. In curious contrast with the swiftly moving α-particles and the artificially accelerated proton streams, and other corpuscular rays, the capacity of the neutron to produce artifical transformations increases, in most cases, with the diminution of its velocity. Doubtless this is connected with the fact that its small velocity enables the particle to stay longer in the immediate neighbourhood of the nucleus and thus to bring about more far-reaching changes.

Induced Radioactivity

It has been established by the work of Fermi and other members of the Italian school that a few collisions of neutrons with hydrogen nuclei (present in either a free or a combined state) are sufficient to establish a sort of thermal equilibrium, so that the neutrons on the average attain, after a few collisions, the averave velocity of hydrogen particles at room-temperature. Whereas the transmutation of elements to stable varieties of other elements had been previously noticed or conjectured, during the bombardment by α-particles, or protons, Curie and Joliot discovered in 1933, that unstable and hitherto unknown isotopes of light elements are produced by the bombardment of light nuclei with α-particles. These unstable nuclei subsequently break up in the same way as ordinary radioactive elements, and phenomena of induced radioactivity are found to obey the same laws, and decay in the same characteristic way as the natural activities of the well-known radio-elements. The identity of these artificial bodies can also be often established by purely chemical methods. In their pioneer work Curie and Joliot discovered that unstable isotopes of nitrogen, silicon, and phosphorus are formed by the bombardment of Be, Al, and Mg: these however emit, curiously enough, positive electrons though the phenomena of disintegration proceeds like the familiar β- ray disintegration of radioactive elements. The discovery that new radioactive elements can be produced by artificial transmutation has given a tremendous impetus to the study of the nuclear reactions. The task of following such nuclear changes is a very much easier one, as sensitive physical apparatus like the Geiger Counter can be utilized to detect and measure such changes. This pioneer work of Curie and Joliot has been greatly extended by Fermi and other workers in Italy. They have observed that slow-moving neutrons are in most cases quite effective in producing similar changes and of generating radioactive isotopes. Extensive work in this line has been done and a large number of new radioactive atoms have been discovered. These however almost always emit β -particles, *i.e.*, ordinary negative electrons instead of positrons.

Deuteron—the Isotope of Hydrogen

With the discovery of deuterons, the isotope of hydrogen, a new kind of corpuscle has been utilized for bombardment of atomic nuclei. Acclerated deuteron streams have been utilized both in England and America and they have proved very much more effective as agents for transmutation than the original proton rays of Cockcroft and Walton. Bombardment by deuterons also produces radioactive bodies and this method has been increasingly used in recent times to study the phenomena of induced radioactivity. The earlier methods of production of new radioactive bodies had to utilize a natural radioactive source for the supply of the suitable bombarding agents; this had necessarily limited the amount of the yield even when the process of canalization was utilized for diminishing the velocity of the neutrons and thereby to bring about an enormous increase of output. The yield of radioactive stuff by the bombardment of deuterons has been naturally very much greater. For example, in a day's exposure a yield of radioactive isotope of sodium has been reported to have been obtained by Livingstone whose activity equals that of 1 gramme of radium. Deuteron bombardment has therefore a great future both in the therapeutical application of radioactivity to medical research as well as in the investigation of nuclear problems.

Whereas bombardment of corpuscles has been almost always utlized for producing nuclear changes, results similar to photo-ionization have been obtained by Chadwick and Goldhaber. By utilizing hard-γ-radiation from ThC these investigators have been able to decompose the hydrogen isotope into neutron and proton. The liberated neutron can be detected by its ability to produce induced radioactivity in suitable elements, or by a properly conducted ionization measurement. By hard X-rays emitted in tubes run at more that 1.5 million-volts pressures, the workers in Berlin have been able to eject neutron streams from beryllium, whose presence has been similarly demonstrated by the generation of radio-iodine in ethyl iodide. These preliminary results have great theoretical significance; Chadwick has been able from a tentative determination of the thresh-hold value of the frequency of the γ-rays necessary to decompose deuterons, determine the mass of the neutron as well as the strength of the binding of the two fundamental particles.

Conclusion

In this lecture I have attempted to give a rapid review of the principle results obtained in the field of nuclear physics in recent times. I shall conclude my remarks by mentioning two remarkable results that have been obtained by the bombardment of thorium and uranium, the two heaviest of the elements with neutrons. I have already observed that the production of new radioactive bodies could, in many cases, be proved by chemical methods; this serves at the same time to determine the chemical properties of the new substance, and its position in the periodic table. By the bombardment of uranium, Fermi originally reported the production of new β-ray emitting elements whose chemical behaviour seems to point to their positions beyond uranium itself in the Mendeljieff's table. This has been corroborated by Meitner. According to these workers the table of elements is artificially extended by this process beyond uranium itself—according to the scheme

$$U^{238}_{92} + n^{1}_{0} \longrightarrow U^{239}_{92} \xrightarrow{\beta} \text{Eka Rhenium } ER^{239}_{93}$$

$$\longrightarrow \text{Eka } Os^{239}_{94} \longrightarrow \text{Eka } Ir^{239}_{95}$$

a truly remarkable result.

By a study of the product of the disintegration of thorium by neutron bombardment Curie has established the existence of a new series of radioactive elements whose mass numbers are in the form 4n + 1. This analogous series of radioactive bodies is not known in nature, and its discovery can be regarded as completing our ideas of radioactive disintegration of heavy elements.

It will be evident from what I have reported above that enormous experimental materials have accumulated in the field of nuclear physics within recent years. Sufficient materials are now available for the theoretical physicists to speculate about the process of atom-building from elementary particles, and the preliminary work in this direction has already begun. The neutron and the proton seem to be the two exclusive constituents of all atomic nuclei. The α-particles can themselves be looked upon us composite bodies, built up again from neutrons and protons.

Though the idea of the elementary atom has undergone revolutionary changes in recent years, in a certain way the progress achieved has been satisfactory, as it has simplified the number of ultimate and fundamental particles to only two, instead of the 92 elements of the older atomic theory. This idea of the evolution of the material world from comparatively few primordial stuffs is not however new. Here, as in other fields of physics, old ideas have returned, renovated in a new garb and clothed with more significance. The quantum theory of photons has to a certain extent resuscitated the ancient corpuscular theory of Newton. The recent developments of the nuclear physics have brought back the old Proutian hypothesis, of the evolution of all elements from one or rather two primordial stuffs—the proton and the neutron.*

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Anomalous Dielectric Constant of Artificial Ionosphere

In a recent communication in *Nature,* Mitra and Roy[1] have pointed out an interesting feature in the ionospheric dispersion formula. They have shown that the formula can yield values of the dielectric constant of an ionized medium greater, equal to, or less than unity depending on the degree of ionization, collisional frequency and the exciting wave frequency. They therefore hold that the value of dielectric constant of an ionized gas greater than unity which has been recorded by many investigators for comparatively large ionizations is only an outcome of the complete dispersion formula. The object of this note is twofold. Taking the ionospheric dispersion formula we shall *first* deduce the conditions under which the dielectric constant may assume values greater or less than unity and *secondly* we shall show that in the experiments where anomalous values of the dielectric constant of an ionized gas have hitherto been obtained, the experimental conditions are such that the dispersion formula alone cannot explain the anomaly.

The dispersion formula is given by

$$\left(\mu - \frac{ick}{p}\right)^2 = 1 + \frac{1}{\alpha + i\beta},$$

where $\alpha = -\frac{mp^2}{4\pi Ne^2}$ and $\beta = \frac{mp\nu}{4\pi Ne^2}$

Separating the real and imaginary parts, we have

$$m^2 - \frac{c^2k^2}{p^2} = 1 + \frac{\alpha}{\alpha^2+\beta^2} \qquad \text{..... (1)}$$

and

$$\frac{2ck}{p}.\mu = \frac{\beta}{\alpha^2+\beta^2} \qquad \text{..... (2)}$$

Thus $\frac{c^2k^2}{p^2} - \mu^2 = \frac{|\alpha|}{\alpha^2+\beta^2} - 1$ so that

$$\mu^2 \overset{>}{\underset{<}{=}} 1 \text{ according as } \frac{c^2k^2}{p^2} \overset{>}{\underset{<}{=}} \frac{|\alpha|}{\alpha^2+\beta^2}$$

which is equivalent to the condition :

$$\mu^2 \overset{>}{\underset{<}{=}} 1 \text{ if } \frac{\mu^2c^2k^2}{p^2} \overset{>}{\underset{<}{=}} \frac{|\alpha|}{\alpha^2+\beta^2},$$

but as $\frac{\mu^2c^2k^2}{p^2} = \frac{\beta^2}{4(\alpha^2+\beta^2)^2}$ from (2)

the condition reduces to —

$$\mu^2 \overset{>}{\underset{<}{=}} 1 \text{ if } \frac{\beta^2}{4(\alpha^2+\beta^2)^2} \overset{>}{\underset{<}{=}} \frac{|\alpha|}{\alpha^2+\beta^2}$$

or

$$\frac{\beta^2}{\alpha^2+\beta^2} \overset{>}{\underset{<}{=}} 4\,|\alpha|.$$

Now the left-hand side is always a proper fraction; therefore, if $4\,|\alpha| > 1$, *i.e.* $|\alpha| > {\cdot}25$ we have then always

$$\frac{\beta^2}{\alpha^2+\beta^2} < 4\,|\alpha| > 1, \; i.e. \; \mu^2 < 1 .$$

If however $|\alpha| < {\cdot}25$, $\mu^2 > 1$ will require the further condition

$$\frac{\beta^2}{\alpha^2+\beta^2} > 4\,|\alpha| \text{ or } \beta^2 > \frac{4|\alpha|\,\alpha^2}{1-4|\alpha|}.$$

Calling $\frac{\pi Ne^2}{m} = p_0^2$ or

$$p_0 = \sqrt{\frac{\pi e^2}{m}}\,\sqrt{N} = 2\cdot8\times10^4\sqrt{N},$$

we have $4\,|\alpha| = \frac{p^2}{p_0^2}$ and we can say that if

$p > p_0 = 2{\cdot}8 \times 10^4 \sqrt{N}$, μ is always less than unity, whereas if $p < 2{\cdot}8 \times 10^4 \sqrt{N}$, or $|\alpha| < {\cdot}25$, for $\mu^2 > 1$, we must have

$$v^2 > \frac{p^4}{p_0^2 - p^2}$$

i..e., the collisional frequency must exceed the value

$$\frac{p^2}{\sqrt{p_0^2 - p^2}}$$

Thus the equations show that even when $p < 2{\cdot}8 \times 10^4 \sqrt{N}$, there exists a critical collisional frequency which has to be executed if μ is to be greater than unity.

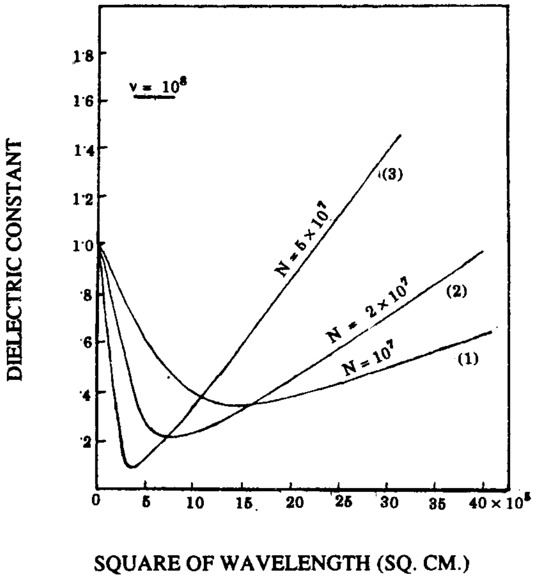

Fig. I

In the experiments recently performed in this laboratory, we have obtained results similar to those of Appleton and Childs[2]. Using a wavelength of about 3·8 metres the dielectric constant of ionized air in a discharge tube has been found to be less than unity when the tube current is small. As the ionization is increased by increasing the tube current, the dielectric constant at first decreases and then increases gradually and even becomes greater than unity. The maximum electron density N in our discharge tube has been within 10^7. Taking $N=10^7$, $p=5 \times 10^8$, we get $|\alpha| = 7{\cdot}8$. This is much greater than ·25 so that the dielectric constant must be less than unity. Even when $N=10^8$, $|\alpha| = {\cdot}78$ and the dielectric constant should be theoretically less than unity.

In Fig. I are plotted the calculated values of the dielectric constants against λ^2 for some definite values of N and v. Curve (2) which corresponds to Mitra and Banerjee's[3] experimental conditions shows a turning point at λ of the order 9m (λ^2 is of the order 8×10^5 sq.cm.) which is very different from the experimental value λ of the order 6m (λ^2 is of the order $3{\cdot}6 \times 10^5$ sq.cm.). In the experiments of Ali Imam and one[4] of us, which have been recently repeated by a different method, the turning point appears unmistakably at λ of the order 4m (λ^2 is of the order $1{\cdot}6 \times 10^5$ sq.cm.). Curve (1) which approximately corresponds to our experimental conditions shows however the turning point at λ of the order 12m (λ^2 is of the order 14×10^5 sq.cm.). It appears extremely doubtful if the disposition of the experimental apparatus can account for such wide discrepancy. It is likely that there must be some other explanation of these experimental results.

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13.11.37

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ON THE TOTAL REFLECTION OF ELECTROMAGNETIC WAVES IN THE IONOSPHERE

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The conditions under which the electromagnetic waves get totally reflected in the upper atmosphere appear still worth a closer investigation. The usual procedure has been to attribute to the medium a refractive index which is calculated according to the classical method of Lorentz from the mechanical equations of motion of electrons. In an absorbing medium, this refractive index is a complex quantity, and depends not only on the properties of the medium but also on the frequency of the waves : as progressive waves through the ionosphere are damped, the co-efficient of damping will also enter into the expression of the refractive index. The usual Appleton-Hartree condition for total reflexion has been deduced by neglecting the damping, as a first approximation, and by putting the refractive index equal to zero. Discussion of the conditions when the damping is not negligible, gets very complicated indeed. Certain workers in Allahabad have suggested the further condition that the group-velocity of the wave-train is zero when total reflection takes place. This is a reasonable hypothesis, but the method by which the group-velocity is calculated neglects damping, an essential factor in the physical process. It appears also that the calculation of the group-velocity of waves in an absorbing medium is not amenable to the ordinary methods unless the damping is negligible, an approximation which, as has been already pointed out, does not seem to agree closely with experience. It seems therefore necessary to change the method of analysis and to take, instead of the equations of Maxwell with a complex refractive-index, the microscopic equations of Lorentz as basis for calculation. The method sketched here is a general one suitable for problems of refractive-index of material media as well as for discussion of propagation of waves in the ionosphere. It seems further amenable to modifications to suit the requirements of modern wave mechanics.

We take the familiar equations of Lorentz

$$\frac{I}{c}\,\frac{\partial E}{\partial t} - \text{curl } H = -\frac{\rho V}{c}\,;\ \frac{I}{c}\,\frac{\partial H}{\partial t} + \text{curl } E = 0.$$

$$\text{div } E = \rho, \quad \text{and} \quad \text{div } H = 0.$$

The density and the convection current, ρ and ρV, can be regarded as decomposable into two parts ρ_+, and ρ_- and $(\rho V)_+$ and $(\rho V)_-$, respectively, which thus takes account of the corpuscular distribution of positive and negative charges. A wave disturbance in the medium can be considered by putting

$$E = E_0+Ee^S,\ H = H_0+He^S;\quad \rho_+ = \rho_+^0+q_+e^S,\ \rho_- = \rho_-^0+q_-e^S,$$

$V_+ = V_+^0+W_+e^S$ and $V_- = V_-^0+W_-e^S$, where the elements with 0 indices correspond to the undisturbed state of the medium. The convection current, and the electric-density become on this assumption equal to

$$\overline{\rho V} = \rho_+V_+ + \rho_-V_- = (\overline{\rho V})_0+\bar{\theta}e^S$$

where $$\bar{\theta} = [\rho_+^0\, W_+ + q_+V_+^0+\rho_-^0W_-+q_-V_-^0) \quad \text{and} \quad \theta_0 = q_++q_- .$$

Thus the components $(\theta_x, \theta_y, \theta_z)$ of the vector $\bar{\theta}$ correspond to the amplitudes of the fluctuating part of the convection current, and θ_0, similarly to the fluctuating part of the electric density due to the disturbance.

The process is a familiar one : When S is a linear function of $x, y, z, t,$ with a complex factor $2\pi\iota$, as is generally chosen for discussing monochromatic wave-propagation, $2\pi\iota\, S$ represents the phase of the disturbance and the surfaces for various values of S, represent the totality of all wave fronts as function of (x, y, z, t). In order to tackle the more general case of a damped wave-train we shall not at first put any limitation on the form of S but shall assume it to be any function whatever, whose differential co-efficients may also have complex values. Removing the contributions on the both side of the equation by the various quantities corresponding to the undisturbed state, we get

$$\frac{1}{c}\frac{\partial}{\partial t}(Ee^S) - \text{curl }(He^S) = -\frac{\theta}{c}\, e^S$$

$$\frac{1}{c}\frac{\partial}{\partial t}(He^S) + \text{curl }(Ee^S) = 0$$

$$\text{div }(Ee^S) = \theta_0 e^S, \text{ and div }(He^S) = 0.$$

We can also verify easily that

$$\text{div }[Ee^S] = [\text{div } E+(\text{grad } S.E)]e^S$$

$$\text{curl }[Ee^S] = [\text{curl } E+\text{grad } S\times E]e^S$$

$$\frac{\partial}{\partial t}\left[\, Ee^S\right] = \left[\frac{dE}{dt}+E.\frac{\partial S}{\partial t}\right] e^S$$

etc.

Under the usual conditions of propagation E, H, representing the amplitudes of the electric and magnetic waves, are slowly varying quantities whose differential co-efficients with regard to time and space co-ordinates can be neglected in comparison with the rapid variations of the function S which represents the phase of the disturbance. We shall therefore put the differential co-efficients of E, H equal to zero, and thus get finally the equations reduced to the following form :—

$$\frac{1}{c}\,\frac{\partial S}{\partial t}\,E-(\text{grad } S\times H)=-\theta/c$$

$$\frac{1}{c}\,\frac{\partial S}{\partial t}\,H+(\text{grad } S\times E)=0$$

$$(\text{grad } S\cdot E)=\theta_0$$
$$(\text{grad } S\cdot H)=0$$

These can be regarded as the characteristic set of the wave-equations of Lorentz, and since there are eight equations, if $\frac{\partial S}{\partial x}$, $\frac{\partial S}{\partial y}$, $\frac{\partial S}{\partial z}$, $\frac{\partial S}{\partial t}$ are taken as unknown, certain compatibility conditions have further to be satisfied which will give us the conditions under which wave-propagation can take place. We proceed to calculate these conditions by the usual method of vector-analysis. We have only to remember that, as the components of vectors may have complex-values, the geometrical interpretation of the various quantities may not be so immediate as in the usual cases : also a relation $A^2=0$, does not mean that the vector vanishes, as the components of the vector A_x, A_y, A_z, need not be zero, when $A_x^2+A_y^2+A_z^2=0$.

We shall call such vectors as singular, and in the problem that we have got before us, considerations of singular-vectors become important.

A few remarks regarding the geometrical interpretation of a singular vector may not be out-of-place; regarding the components of a singular vector A, as decomposable into real and imaginary parts, we can always choose a real direction (λ, μ, ν) which is perpendicular to a complex vector, *viz.*, $(A_x+A_x^{*}\iota, \ldots)$

i.e., such that $$\lambda A_x+\mu A_y+\nu A_z=0$$
$$\lambda A_x^{*}+\mu A_y^{*}+\nu A_z^{*}=0$$

i.e., an imaginary vector A, may be regarded as defining a real plane.

If co-ordinates are chosen so as to represent this plane as the plane of (xy), a singular vector A, i.e., whose $A^2=0$, can be represented as

$(A, \iota A, 0)$ where A, may be a complex quantity, i.*e*., A is of the form$[\iota\lambda+\iota\mu, -\mu+\iota\lambda, 0]$. Any vector perpendicular to a singular vector, will have in this co-ordinate system, the general form $(P, \iota P, Q)$ where P and Q may be complex quantities. A non-singular vector, C perpendicular to A, will have $C^2\neq 0$, *i.e.*, $Q\neq 0$; a singular vector perpendicular to A, will also necessarily have $Q=0$, *i.e.*, if A and B are two singular vectors, both conditions $(A\cdot B)=0$, and $(A\times B)=0$, may be satisfied at the same time and all singular vectors perpendicular to one another may be regarded as lying in one real plane.

If A and B are two perpendicular vectors of which one is singular while the other is not, any vector which is perpendicular to both A and B, must necessarily be singular, and will lie in the plane of A. This proposition follows immediately by writing out the general form of A and B in the manner explained above, and trying to write down the general form of C which satisfies the orthogonality condition. These preliminary remarks will be useful in discussing the singular case of total reflection, as we shall see later on. The other two relations in vector analysis which we shall have to use pretty frequently are

$$A.(B\times C) = B.(C\times A) = -B.(A\times C) = \text{etc.} \qquad \dots \quad \text{(I)}$$

and

$$A\times(B\times C) = B(AC)-C(AB). \qquad \dots \quad \text{(II)}$$

These have unrestricted validity even when the components of vectors have complex values, as they are formal identities, involving only re-arrangement of terms.

We shall at first deduce certain general conclusions from our equations

$$\frac{1}{c}\frac{\partial S}{\partial t}E-(\text{grad } S\times H) = -\theta/c \qquad \dots \quad (1)$$

$$\frac{1}{c}\frac{\partial S}{\partial t}H+(\text{grad } S\times E) = 0 \qquad \dots \quad (2)$$

$$(\text{grad } S\cdot E) = \theta_0 \qquad \dots \quad (3)$$

$$(\text{grad } S\cdot H) = 0. \qquad \dots \quad (4)$$

Scalar multiplication with grad S of (1) gives, taking account of (3),

$$\frac{1}{c}\left[\frac{\partial S}{\partial t}\theta_0+(\theta \text{ grad } S)\right] = 0, \qquad \dots \quad \text{(A)}$$

a relation which expresses the conservation of charge in the disturbed state of the medium.

Vector multiplication by grad S of (2) gives

$$\frac{1}{c}\frac{\partial S}{\partial t}(\text{grad } S\times H)+\text{grad } S\times(\text{grad } S\times E) = 0\,;$$

taking account of (1) we get

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad } S)^2\right]E+\frac{\theta}{c^2}\frac{\partial S}{\partial t}+\theta_0 \text{ grad } S=0. \qquad \ldots \text{ (B)}$$

Scalar multiplication of (2) with E gives $(EH)=0$ ... (C)

Scalar multiplication (1) with H gives $(\theta H)=0$ ... (D)

Relations (C), (D) and (4) show that the vectors E, θ, and grad S, are all in one plane perpendicular to H ; also (1) shows that grad S is perpendicular to $\frac{\partial S}{\partial t}E+\theta$. This vector can be regarded as representing the rate of change of induction, or $\frac{dD}{dt}$ of Maxwell's equation; thus H and D can be regarded as lying in the plane perpendicular to grad S while E and θ separately do not lie in it.

The arrangement of the various vectors thus agrees with our usual ideas about electromagnetic waves.

ENERGY AND MOMENTUM RELATIONS IN THE WAVE FIELD

Scalar multiplication of (1) and (2) by E and H respectively give

$$\left[\frac{1}{c}\frac{\partial S}{\partial t}E^2+\frac{(E\theta)}{c}\right]=E\cdot(\text{grad } S\times H)=-\text{grad } S.\ [E\times H] \qquad \ldots \text{ (E)}$$

$$\frac{1}{c}\frac{\partial S}{\partial t}H^2=-\text{ grad } S.\ [E\times H]\ ; \qquad \ldots\ (F)$$

addition and subtraction of (E) and (F) give two further important relations

$$\frac{1}{c}\frac{\partial S}{\partial t}\frac{[E^2+H^2]}{2}+\frac{(E\theta)}{2c}=-\text{ grad } S.\ [E\times H] \qquad \ldots\ (G)$$

$$\frac{1}{c}\frac{\partial S}{\partial t}(E^2-H^2)=-\frac{(E\theta)}{c}. \qquad \ldots\ (H)$$

Also vector multiplication of (1), and (2) by H and E similarly gives

$$\frac{1}{c}\frac{\partial S}{\partial t}(E\times H)+H\times(\text{grad } S\times H)=-\frac{\theta\times H}{c}$$

or

$$\frac{1}{c}\frac{\partial S}{\partial t}(E\times H)+H^2\text{ grad } S=-\frac{\theta\times H}{c}\ ; \qquad \ldots \text{ (K)}$$

also $$\frac{1}{c}\frac{\partial S}{\partial t}(E\times H)+E^2\,\text{grad}\,S-E\,(\text{grad}\,S.E)=0$$

or $$\frac{1}{c}\frac{\partial S}{\partial t}\,(E\times H)+E^2\,\text{grad}\,S=\theta_0 E. \qquad \ldots \quad \text{(L)}$$

Subtracting (L) from (K) we have

$$\text{grad}\,S\,(H^2-E^2)=-\left[\,\theta_0 E+\frac{\theta\times H}{c}\right]. \qquad \ldots \quad \text{(M)}$$

Multiplying (K) and (L) by grad S we have the further relation

$$\frac{1}{c}\frac{\partial S}{\partial t}\,\text{grad}\,S.(E\times H)+\text{grad}\,S.\,\frac{(\theta\times H)}{c}=-H^2\,(\text{grad}\,S)^2$$

$$\frac{1}{c}\frac{\partial S}{\partial t}\,\text{grad}\,S.\,(E\times H)=\theta_0^2-(\text{grad}\,S)^2E^2\,;$$

but $$\text{grad}\,S.\,(\theta\times H)=-\,\theta.\,(\text{grad}\,S\times H)=-\,\frac{1}{c}\left(\theta^2+\frac{\partial S}{\partial t}\,(E\theta)\right)$$

or $$\frac{1}{c}\,\text{grad}\,S.\,(\theta\times H)=-\left[\frac{\theta^2}{c^2}+\frac{1}{c^2}\frac{\partial S}{\partial t}\,(E\theta)\right].$$

These two relations thus become

$$\frac{1}{c}\frac{\partial S}{\partial t}\,\text{grad}\,S.\,(E\times H)=\frac{\theta^2}{c^2}+\frac{1}{c^2}\frac{\partial S}{\partial t}\,(E\theta)-(\text{grad}\,S)^2H^2$$

$$\frac{1}{c}\frac{\partial S}{\partial t}\,\text{grad}\,S.\,(E\times H)=-(\text{grad}\,S)^2E^2+\theta_0^2.$$

From these we get

$$\frac{\theta^2}{c^2}-\theta_0^2+\frac{1}{c^2}\frac{\partial S}{\partial t}\,(E\theta)-(\text{grad}\,S)^2(H^2-E^2)=0,$$

or remembering (H) we see that

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad}\,S)^2\,\right](H^2-E^2)=S\,\theta_0^2-\frac{\theta^2}{c^2}. \qquad \ldots \quad \text{(P)}$$

Also $(\text{grad}\,S)^2H^2=(\text{grad}\,S.H)^2+(\text{grad}\,S\times H)^2$

$$=\frac{1}{c^2}\left[\left(\frac{\partial S}{\partial t}\right)^2E^2+2\,\frac{\partial S}{\partial t}\,(E\theta)+\theta^2\right];$$

we get

$$\frac{1}{c}\frac{\partial S}{\partial t}\,\text{grad}\,S\,.\,(E\times H)=-\frac{1}{c^2}\frac{\partial S}{\partial t}\left[E^2\frac{\partial S}{\partial t}+(E\theta)\right]$$

$$=\theta_0^2-(\text{grad}\,S)^2E^2, \qquad \dots \quad (Q)$$

which can be written in the form

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad}\,S)^2\right]E^2+\frac{(E\theta)}{c^2}\frac{\partial S}{\partial t}+\theta_0^2=0. \qquad \dots \quad (\mathrm{Q})$$

This equation could have been immediately deduced from (B) by taking scalar multiplication with E.

Equation (B) :

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad}\,S)^2\right]E+\frac{\theta}{c^2}\frac{\partial S}{\partial t}+\text{grad}\,S(\text{grad}\,S\cdot E)=0$$

gives us generally the mutual disposition of E, θ and grad S when wave-propagation is possible, whereas the equation (Q) written in the form

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad}\,S)^2\right]+\frac{1}{c^2}\frac{\partial S}{\partial t}\frac{(E\theta)}{E^2}+\frac{(\text{grad}\,S\cdot E)}{E^2}=0 \qquad \dots \quad (\mathrm{Q})$$

may be regarded as the equation for the refractive index, determining the velocity of propagation in any direction prescribed by grad S.

THE NOTION OF THE REFRACTIVE-INDEX

A surface $S(x, y, z, t) = C$ may be regarded as moving perpendicular to itself with a velocity given by

$$W=\frac{\dfrac{\partial S}{\partial t}}{|\text{grad}\,S|}\ ;$$

the refractive index can be defined as $\dfrac{c}{W}=\mu=\dfrac{|\text{grad}\,S|}{\dfrac{1}{c}\dfrac{\partial S}{\partial t}}$, agreeing with our usual definition, which however can be complex in the general case.

Introducing μ, equation (Q) can be written in the form

$$(1-\mu^2)+\frac{(E\theta)}{E^2}\left|\frac{\partial S}{\partial t}+\frac{c^2(\text{grad } S\cdot E)^2}{E^2}\right|\left(\frac{\partial S}{\partial t}\right)^2=0.$$

ON RELATION BETWEEN E AND θ

Early in the paper we have expressed θ, in terms of motion of the positive and negative charges, namely,

$$\theta=[\rho_0^+W_0^++q^+V_0^++\rho_0^-W^-+q^-V_0^-].$$

These involve in general both the original distribution of the charges, their velocities as well as the amplitudes of velocities, and densities of positive and negative electricity due to the disturbance in the medium.

From general considerations we can express the relation of θ, and E in the following way.

We have seen that during propagation of disturbance both θ and E lie in the plane perpendicular to H. Therefore, θ can be decomposed into two components, one along E and another perpendicular to E, in this plane, *i.e.*, in the direction of $(E\times H)$.

θ thus can be put as equal to $\alpha E+\beta\,\dfrac{[E\times H]}{|E|}$

$$\because\quad (\theta E)=\alpha E^2,\qquad \theta\times E=\frac{\beta E^2}{|E|}\,H;$$

also from equations of wave--propagation

$$\theta\times E=cH\,(\text{grad } S\cdot E)\ \text{ or }\ \beta^2/c^2=\theta_0^2/E^2.$$

Writing therefore

$$\theta=\alpha E+\beta\,\frac{[E\times H]}{|E|}$$

we see that

$$\alpha=\frac{(E\theta)}{E^2},\ \beta=\frac{C\theta_0}{|E|},$$

so that the refractive-index equation can be expressed in terms of α and β in the form

$$(1-\mu^2)+\frac{\alpha}{\partial S/\partial t}+\frac{\beta^2}{(\partial S/\partial t)^2}=0.$$

Further.

$$\theta^2 = \alpha^2 E^2 + \beta^2 H^2.$$

As the vectors E and H are perpendicular to one another it follows that α and β are two co-efficients which stand in very simple relation to the total polarisation current; α and β however are not constants but may involve grad S as well as $\frac{\partial S}{\partial t}$, *i.e.*, may be function of direction and frequency of the waves.

CONDITION FOR TOTAL REFLECTION

It is well-known that for electro-magnetic waves in vacuum we have $E^2 - H^2 = 0$ as well as $(E \cdot H) = 0$. For waves in material media however $E^2 - H^2 \neq 0$, but $(EH) = 0$ remains still valid. The ratio of the electric and magnetic vectors depends generally upon the properties of the medium as well as on the frequency of the wave and we can express this fact by the relations (H) and (M)

which are
$$\frac{1}{c}\frac{\partial S}{\partial t}(E^2 - H^2) = -\frac{(E\theta)}{c}$$

$$\text{grad } S(H^2 - E^2) = -F = -\left(E\theta_0 + \frac{\theta \times H}{c}\right)$$

The ratio of the amplitudes of the vectors, H^2/E^2, can therefore be expressed by the following relations :

$$\frac{\partial S}{\partial t} + \frac{(E\theta)}{E^2} = \frac{H^2}{E^2}\frac{\partial S}{\partial t}$$

$$\text{grad } S - \frac{F}{E^2} = \frac{H^2}{E^2} \text{ grad } S,$$

Depending upon the property of the medium and the direction of disposition of the vector E, $\frac{(E\theta)}{E^2}$ will generally vary with $\frac{\partial S}{\partial t}$ as well as with grad S. If, however, $\frac{\partial S}{\partial t}$ tends to such a value that $\frac{\partial S}{\partial t} + \frac{E\theta}{c^2}. = 0$, it follows that $H^2 = 0$ $(E \neq 0)$, independent of the amplitude of the wave traversing the medium; the disturbance in such a case loses its wave-character, so that we can say the wave cannot penetrate beyond that region and gets totally reflected. We shall therefore take this condition $\frac{H^2}{E^2} = 0$ or an equivalent condition as characterising the condition of total reflection and we can also correlate it with another physical idea.

We have also proved the following relation (F)

$$\frac{1}{c}\frac{\partial S}{\partial t}H^2 = -\text{grad } S.\,[E\times H]$$

and also

$$\frac{1}{c}\frac{\partial S}{\partial t}\left[\frac{E^2}{2}+\frac{H^2}{2}\right]+\frac{E\theta}{2c} = -\text{grad } S.\,(E\times H).$$

grad $S.(E\times H)$ may thus be regarded as proportional to the flow of energy across the surface $S =$ constant, so that grad $S.(E\times H) = 0$ may be interpreted to mean that the flow of energy across the wave-front vanishes. This agrees with our usual idea of total-reflection at a boundary of two media, where the energy flows along the boundary interface in the limiting case.

We have also seen that

$$\frac{|\text{grad } S|}{\frac{1}{c}\frac{\partial S}{\partial t}} = \mu = \text{the refractive index of the medium.}$$

From (2) we have $\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 H^2 = (\text{grad } S\times E)^2 = E^2(\text{grad } S)^2-(\text{grad } S.E)^2$

$$= -\frac{1}{c}\frac{\partial S}{\partial t}\,\text{grad } S.[E\times H),$$

we have

$$\frac{H^2}{E^2} = \frac{(\text{grad } S)^2}{\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2} - \frac{(\text{grad } S.E)^2}{E^2\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2} = -\frac{\text{grad } S.[E\times H]}{\frac{1}{c}\frac{\partial S}{\partial t}E^2} = \frac{H.(\text{grad } S\times E]}{\frac{1}{c}\frac{\partial S}{\partial t}E^2}.$$

In case (grad S$\cdot E$) = 0, we see that the condition $\frac{H^2}{E^2} = 0$ is equivalent to the condition $\mu^2 = 0$, as has been assumed by Appleton to characterise the case of total reflection. If we now examine the condition (B) which gives the relation between the vectors E, θ and grad S, the condition $\theta = 0$ will mean that the wave is propagated under such conditions that the vector E is parallel to θ.

Multiplying (1) with E, we have

$$E\times\theta = cE\times(\text{grad } S\times H)$$

$$= -\,cH(\text{grad S}\cdot E),$$

so that (grad $S\cdot E$) = 0 will mean $E\times\theta = 0$, *i.e.*, E is paralled to θ.

We thus see that if the wave is such that E is || to θ the condition of total reflection $H = 0$, or grad $S.(E \times H) = 0$ is equivalent to the assumption $\mu = 0$; in the general case of wave propagation when E is not parallel to θ we would expect on the other hand the condition $H^2 = 0$ as giving a new result.

$H^2 = 0$, or grad $S.(E \times H) = 0$, may be regarded as equivalent to the assumption that the group-velocity vanishes. The condition of total-reflection postulated above is obviously suitable for a train of damped waves, because our analysis is general and the condition of non-penetrability has been deduced independent of all ideas of frequency.

We shall now apply the results of our analysis in the particular case of the ionosphere.

The contribution of the positive charges, as well as of the bound electrons in θ, may be easily seen to be negligible in comparison with the contribution of the free electrons.

We can assume for the ionosphere

$$\theta = [\rho_0^- W^- + q^- V_0^-]$$

where these quantities refer only to the free-electrons present. If we average over a volume containing a large number of electrons, but small compared with the wave-length of the disturbance,

$$\rho_0^- = Ne, \quad \theta = \overline{NeW} \ ;$$

the average value of the second term $q\bar{V}_0$ may be regarded as zero as the initial velocities of the charges may be regarded as distributed chaotically in all directions.

Writing the equation of motion of a free electron after Lorentz as

$$m\ddot{x} + gx = e\left[E_x + \frac{v_y h_z - v_z h_y}{c}\right], \text{ etc.}$$

where (h_x, h_y, h_z) may be regarded as the components of a steady magnetic force, we have, dividing by m and multiplying by $\rho = Ne$, and averaging, the following equation

$$\dot{\theta} + \frac{g}{m}\theta = \frac{Ne^2}{m} E + \left(\theta \times \frac{eh}{cm}\right)$$

where $\qquad (\theta_x = \overline{Nev_x}, \text{ etc...}).$

Remembering that for an oscillatory disturbance $\theta = \theta \dfrac{\partial S}{\partial t} e^S$ as before also that the other terms in the equation varies at e^S, the equation can be written as

$$\left(\frac{\partial S}{\partial t}+\frac{g}{m}\right)\theta=\frac{Ne^2}{m}E+\left(\theta\times\frac{eh}{cm}\right) \qquad \ldots \quad (Z)$$

cancelling out e^S. $\frac{g}{m}=\nu$ is usually called the collision frequency of the electrons. We shall use this equation (Z) to calculate $\frac{(E\theta)}{E^2}$ or $\frac{\theta}{E}$.

Case I : This can be regarded as decomposable into two subcases.

(*a*) When $\frac{\theta\times eh}{cm}=0$, *i.e.*, θ is parallel to the external magnetic field, equation (Z) shows that θ is then parallel to E

and
$$\frac{E\theta}{E^2}=\frac{\theta}{E}=\frac{Ne^2}{m}\left(\frac{\partial S}{\partial t}+\nu\right).$$

The equation for determining the critical frequency for total reflection becomes

$$\frac{\partial S}{\partial t}+\frac{Ne^2}{m}\Big/\left(\frac{\partial S}{\partial t}+\nu\right)=0$$

or
$$\left(\frac{\partial S}{\partial t}\right)^2+\nu\left(\frac{\partial S}{\partial t}\right)+\frac{Ne^2}{m}=0$$

or
$$\frac{\partial S}{\partial t}=-\frac{\nu}{2}\pm\iota\sqrt{\frac{Ne^2}{m}-\nu^2/4}$$

This determines the dependence of the critical frequency on the ionic density; the train totally reflected is therefore of the form

$$e^{-\nu/2t}\,e^{\pm\iota t\sqrt{Ne^2/m-\nu^2/4}}$$

or
$$w_c^2=Ne^2/m-\nu^2/4$$

and the damping co-efficient is $\nu/2$.

Case II : When $\theta \| E$ and $\theta^2=0$

that is the vector θ is singular, $\theta^2=0$ but $\theta\neq 0$

a vector $\left(\theta\times\frac{eh}{cm}\right)$ will represent a vector parallel to θ, when h is non-singular and perpendicular to θ according to our former remarks about singular vectors. Choosing the direction of h, as the direction of Z, we can write the relation

$$\left(\frac{\partial S}{\partial t}+\nu\right)\theta=\frac{Ne^2}{m}E+\left(\theta\times\frac{eh}{cm}\right)$$

in the form

$$\left(\frac{\partial S}{\partial t}+\nu\right)\theta_x=\frac{Ne^2}{m}E_x+\frac{eh}{cm}\theta_y$$

$$\left(\frac{\partial S}{\partial t}+\nu\right)\theta_y=\frac{Ne^2}{m}E_y-\frac{eh}{cm}\theta_x$$

or $$\left(\frac{\partial S}{\partial t}+\nu\pm\iota\frac{eh}{cm}\right)(\theta_x+\iota\theta_y)=\frac{Ne^2}{m}(E_x\pm\iota E_y)$$

$\frac{\theta}{E}$ becomes equal to $\dfrac{Ne^2/m}{\left[\frac{\partial S}{\partial t}+\nu\pm\iota\frac{eh}{cm}\right]}$,

according as θ and E are of the form $(A, \mp\iota A, 0)$.

The condition of total reflection now becomes, from (B),

$$\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\pm\iota\frac{eh}{cm}\right)+\frac{Ne^2}{m}=0;$$

calling $$\frac{\partial S}{\partial t}+\nu/2\pm\iota\frac{eh}{2cm}=x$$

we have $$x^2+\frac{Ne}{m}-\nu^2/4+\frac{e^2h^2}{4c^2m^2}\pm\frac{eh\iota\nu}{2cm}=0$$

Solving the above equation, we find that the two reflected waves are circularly polarised and the critical frequencies are

$$\omega_{c\pm}=\sqrt{\frac{Ne^2}{m}-\nu^2/4+\frac{e^2h^2}{4c^2m^2}}\mp\frac{eh}{2\,cm}$$

$$T_{c\pm}=\nu/2\left[1\pm\frac{eh}{2cm\omega'}\right]\text{ where }\omega'=\sqrt{\frac{Ne^2}{m}-\nu^2/4+\frac{e^2h^2}{4c^2m^2}}$$

So that when we have the critical frequency $=\frac{eh}{cm}$, *i.e.*, $\omega'=\frac{eh}{2cm}$, one of the circularly polarised wave will be suppressed, and the other will be reflected with undiminished intensity : as this will necessarily mean $\frac{Ne^2}{m}=\nu^2/4$ the reflection of plane-polarised wave, discussed above in case I, will not occur as the condition of total reflection gives $\omega_c = 0$, if $\frac{Ne^2}{m} = \nu^2/4$.

These two cases correspond to the cases discussed by Appleton by putting $\mu = 0$ and the result agrees with the usual formula, when the collision-frequency is neglected.
Case III. When $\theta \neq E$:

We shall now calculate the value of $\frac{(E\theta)}{E^2}$ in the general case.

We have

$$\left(\frac{\partial S}{\partial t}+\nu\right)\theta = \frac{Ne^2}{m}E+\left(\theta\times\frac{eh}{cm}\right).$$

Scalar multiplication with E gives

$$\left(\frac{\partial S}{\partial t}+\nu\right)(E\,\theta) = \frac{Ne^2}{m}E^2-\theta\left[E\times\frac{eh}{cm}\right];$$

also vector multiplication with E gives

$$\left(\frac{\partial S}{\partial t}+\nu\right)(E\times\theta) = \theta.\left(E\cdot\frac{eh}{cm}\right)-\frac{eh}{cm}(E\theta).$$

Scalar multiplication by $\frac{eh}{cm}$ of the above relation gives

$$\left(\frac{\partial S}{\partial t}+\nu\right)\frac{eh}{cm}.(E\times\theta) = \left(\frac{eh}{cm}.\theta\right)\left(E.\frac{eh}{cm}\right)-\left(\frac{eh}{cm}\right)^2(E\theta).$$

Again, scalar multiplication of the original equation with $\frac{eh}{cm}$ gives

$$\left(\frac{\partial S}{\partial t}+\nu\right)\left(\frac{eh}{cm}\theta\right) = \frac{Ne^2}{m}\left(E\frac{eh}{cm}\right).$$

Combining (a), (b) and (c) we can easily deduce

$$\frac{(E\theta)\Big[\Big(\frac{\partial S}{\partial t}\Big)+\nu\Big]^2+\Big(\frac{eh}{cm}\Big)^2\Big]}{\left(\frac{\partial S}{\partial t}+\nu\right)} = \frac{\frac{Ne^2}{m}(E^2)\Big[\Big(\frac{\partial S}{\partial t}+\nu\Big)^2+\frac{1}{E^2}\Big(E.\frac{eh}{cm}\Big)^2\Big]}{\left(\frac{\partial S}{\partial t}+\nu\right)^2}.$$

or
$$\frac{(E\theta)}{E^2}=\frac{Ne^2}{m}\,\frac{\left[\left(\frac{\partial S}{\partial t}+\nu\right)^2+\frac{1}{E^2}\left(E_1\frac{eh}{cm}\right)^2\right]}{\left(\frac{\partial S}{\partial t}+\nu\right)\left[\left(\frac{\partial S}{\partial t}+\nu\right)^2+\left(\frac{eh}{cm}\right)^2\right]}$$

when E and θ are non-singular vectors.

Now if we consider the case of total reflection as defined by grad $S\cdot[E\times H]$ $= 0$, [Q] gives $\frac{\partial S}{\partial t}=-\frac{(E\theta)}{E^2}$ as well as $\theta_0^2-(\text{grad } S)^2E=0$.

Now also grad $S\cdot[E\times H]$ can be written in the form $H(\text{grad } S\times E)$, we see that grad $S\|E$

and
$$\frac{1}{E^2}\left(E.\frac{eh}{cm}\right)^2=\frac{1}{(\text{grad } S)^2}\left(\text{grad } S\,\frac{eh}{cm}\right)^2$$

$\frac{1}{E^2}\left(E_1\frac{eh}{cm}\right)^2=\left(\frac{eh}{cm}\right)^2\cos{}^2\alpha$ if grad S is a vector whose components have real ratios.

The relation $\frac{\partial S}{\partial t}+\frac{(E\theta)}{E^2}=0$ becomes

$$\frac{\partial S}{\partial t}+\frac{\frac{Ne^2}{m}\left[\left(\frac{\partial S}{\partial t}+\nu\right)^2+\frac{1}{E^2}\left(E\,\frac{eh}{cm}\right)^2\right]}{\left(\frac{\partial S}{\partial t}+\nu\right)\left[\left(\frac{\partial S}{\partial t}+\nu\right)^2+\left(\frac{eh}{cm}\right)^2\right]}=0$$

If the disturbance is plane-polarised and the collision frequency is small we can put

$$\nu=0,\ \frac{1}{E^2}\left(E_1\frac{eh}{cm}\right)^2=\cos^2\alpha\left(\frac{eh}{cm}\right)^2$$

where α is the angle between the direction of propagation and the magnetic lines. The formula becomes, putting $\frac{\partial S}{\partial t}=\iota\,\omega_c$

$$\omega_c^2=\frac{\frac{Ne^2}{m}\left[\omega_c^2-\left(\frac{eh}{cm}\right)^2\cos^2\alpha\right]}{\omega_c^2-\left(\frac{eh}{cm}\right)^2}$$

agreeing with the formula supposed to be derived by putting group-velocity = zero by Bajpai and Mathur.[1]

We shall have to remember however that E may be a complex-vector in the general case, any complex vector $E = [E_x + \iota E_x$, etc.], may be regarded as perpendicular to a real direction, and choosing this direction as the direction of Z, we can choose as components of the complex vector, $(E_x, \iota E_y, 0)$ where x, and y are certain special directions and the external field is regarded as having components (h_x, h_y, h_z) in these special directions.

$$\left(E_1 \frac{eh}{cm}\right) \text{ then becomes equal to } \frac{e}{cm}(E_x h_x + \iota E_y h_y)$$

$$E^2 = E_x^2 - E_y^2 \neq 0$$

$$\frac{1}{E^2}\left(E_1 \frac{eh}{cm}\right)^2 = \frac{e^2}{c^2m^2}\left[\frac{E_x^2h_x^2 - E_y^2h_y^2}{E_x^2 - E_y^2} + 2\iota h_x h_y \frac{E_x E_y}{E_x^2 - E_y^2}\right]$$

The frequency equation for the elliptically polarised beams would be

$$0 = \frac{\partial S}{\partial t} + \frac{Ne^2}{m}\,\frac{\left[\left(\frac{\partial S}{\partial t} + \nu\right)^2 + \frac{e^2}{c^2m^2}\left[\frac{E_x^2h_x^2 - E_y h_y^2}{E_x^2 - E_y^2} + \frac{2\iota E_x E_y}{E_x^2 - E_y^2} h_x h_y\right]\right]}{\left(\frac{\partial S}{\partial t} + \nu\right)\left[\left(\frac{\partial S}{\partial t} + \nu\right)^2 + \frac{e^2h^2}{c^2m^2}\right]};$$

when the collision frequency is neglected this similarly becomes

$$\omega^2 = \frac{Ne^2}{m}\,\frac{\left[\omega^2 - \frac{e^2}{c^2m^2}\left[\frac{E_x^2h_x^2 - E_y^2h_y^2}{E_x^2 - E_y^2}\right]\right]}{\omega^2 - e^2h^2/c^2m^2} \text{ when } \frac{E_x E_y}{E_x^2 - E_y^2} h_x h_y = 0.$$

This shows that the two cases will be either $h_x = 0$, or $h_y = 0$

$$\omega^2 = \frac{\frac{Ne^2}{m}\left[\omega^2 - \frac{e^2h_x^2E_x^2}{E_x^2 - E_y^2}\right]}{\omega^2 - \frac{e^2h^2}{c^2m^2}}, \text{ or } \frac{\frac{Ne^2}{m}\Big[\omega^2 + \Big[\frac{e^2}{c^2m^2}\frac{h_y^2E_y^2}{(E_x^2 - E_y^2)}\Big]}{\omega^2 - \frac{e^2h^2}{c^2m^2}},.$$

We have therefore

or
$$\omega^2 = \frac{\frac{Ne^2}{m}\left[\omega^2 + \frac{e^2h^2}{c^2m^2}\frac{E_y^2}{E_x^2 - E_y^2}\cos^2\alpha\right]}{\omega^2 - \frac{e^2h^2}{c^2m^2}}$$

when h is $\perp$ to the major axis,

$$\omega^2 = \frac{\frac{Ne^2}{m}\left[\omega^2 - \frac{e^2h^2}{c^2m^2}\,\frac{E_x^2\cos^2\alpha}{E_x^2 - E_y^2}\right]}{\omega^2 - \frac{e^2h^2}{c^2m^2}}$$

when h is $\perp$ to the minor axis.

THE PROPAGATION OF THE DISTURBANCE

We shall analyse now the relation (B), *i.e.*,

$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\text{grad } S)^2\right]E + \frac{\theta}{c^2}\,\frac{\partial S}{\partial t}\,\theta_0 \,\text{grad } S = 0$$

which may be regarded as prescribing the relation between the three vectors E, θ and grad S, when propagation takes place.

For this purpose, we shall have to consider the relation (A) which expresses the conservation of charge, *i.e.*,

$$\frac{1}{c}\left[\frac{\partial S}{\partial t}\,\theta_0^2 + (\theta \,\text{grad } S)\right] = 0,$$

and the relation (Z), *i.e.*,

$$\left(\frac{\partial S}{\partial t} + \nu\right)\theta = \frac{Ne^2}{m}E + \left(\theta \times \frac{eh}{cm}\right).$$

We can utilise (A) and (Z) to eliminate E and θ_0 from (B) and thus get

$$\frac{\theta}{c^2}\,\frac{\partial S}{\partial t} + \frac{\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \text{grad}^2 S\right]\left[\left(\frac{\partial S}{\partial \nu} + \nu\right)\theta - \left(\theta \times \frac{eh}{cm}\right)\right]}{\frac{Ne^2}{m}} = \frac{\text{grad } S\,(\theta.\,\text{grad } S)}{\frac{\partial S}{\partial t}}$$

or by re-arranging terms,

$$\left\{\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - \text{grad}^2 S\right]\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t} + \nu\right) + \frac{Ne^2}{mc^2}\left(\frac{\partial S}{\partial t}\right)^2\right\}\theta$$

$$-\frac{\partial S}{\partial t}\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\text{grad}^2 S)\right]\left(\theta \times \frac{eh}{cm}\right) = \frac{Ne^2}{m}\,\text{grad } S\,(\theta.\,\text{grad } S). \quad \text{(R)}$$

This vector equation is equivalent to three homogeneous linear equations in $(\theta_x, \theta_y, \theta_z)$, whose co-efficients involve the differential co-efficients of S and the collision-frequency; when the relation is satisfied for non-zero values of $(\theta_x, \theta_y, \theta_z)$ the discriminant of the simultaneous equation must vanish. This will give a relation between the differential co-efficients of S alone, where the characteristic numbers of the ionospheric layer, *i.e.*, N, h and ν, will also be involved, and may be regarded as giving the equation of the characteristic surface, corresponding to the Ei-konal equation of Hamilton for a material medium

or equation
$$\frac{1}{c^2}\left[\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad } S)^2\right]=0 \qquad \text{for vacuum.}$$

It is better for perspicacity to introduce certain abbreviations at this stage; we shall put

$$g(S)=\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\text{grad}^2 S$$

also
$$\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\text{grad}^2 S\right]\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)+\frac{Ne^2}{mc^2}\left(\frac{\partial S}{\partial t}\right)^2 \text{is put}$$

$$=g(S)\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)+\frac{Ne^2}{m}\left[g(S)+\text{grad}^2\, S\right]=L(S)$$

$$=g(S)\,p(S)+\frac{Ne^2}{m}(\text{grad } S)^2=\frac{p(S)}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)\text{grad}^2\, S$$

where
$$p(S)=\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)+\frac{Ne^2}{m}.$$

The vector equation (R) is thus written

$$L(S)\,\theta-g(S)\frac{\partial S}{\partial t}\left(\theta\times\frac{eh}{cm}\right)=\frac{Ne^2}{m}\text{grad } S(\theta\cdot\text{grad } S).$$

Scalar multiplication with θ gives

$$L(S)\theta^2=\frac{Ne^2}{m}(\theta\cdot\text{grad } S)^2,$$

so that $(\text{grad } S\cdot\theta)=0$, either when $\theta^2=0$ or when $L(S)=0$, so that the vector equation becomes simplified, in both of these cases.

$E \parallel \theta$

Case I. When $L(S)=0$, $\theta^2\neq 0$.

The vector equation (R) is satisfied only when

$$\left(\theta\times\frac{eh}{cm}\right)=0,\ \text{or}\ g(S)=0,\ \text{or}\ \frac{\partial S}{\partial t}=0.$$

We reject $\frac{\partial S}{\partial t}=0$, as we are considering wave propagation. We can easily see that $g(S)=0$ is a singular case, which is not relevant to the problem before us, as $g(S)=0$ will lead to the vanishing of the vector θ.

θ is parallel to h, the external magnetic field.

The characteristic equation thus becomes

$$\frac{p(S)}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\left(\frac{\partial S}{\partial t}+\nu\right)\frac{\partial S}{\partial t}\,\text{grad}^2 S=0 \qquad \ldots\ (1)$$

or

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2=\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2\frac{\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)+\frac{Ne^2}{m}}{\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)}$$

or putting

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2=\frac{\mu}{c^2}\left(\frac{\partial S}{\partial t}\right)^2$$

$$\mu^2=1+\frac{Ne^2}{m}\Big/\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)$$

which agrees with the ordinary expression for the refractive index for plane polarised ordinary ray. The singular solution of the equation (1) when $p(S)$ and $|\text{grad}\ S|=0$ gives the case of total reflection.

Case II. When $\theta^2=0$

$$(\theta\ \text{grad}\ S)=0.$$

The vector equation (R) can be satisfied only when

$$L(S)\theta-g(S)\frac{\partial S}{\partial t}\left(\theta\times\frac{eh}{cm}\right)=0$$

i.e., when $\left(\theta\times\frac{eh}{cm}\right)$ becomes parallel to θ.

This singular case, already discussed, means that θ is perpendicular to h, and choosing a new set of perpendicular axis, with h in the direction of Z, we see $L(S)$, $p(S)$, $g(S)$ are invariants for orthogonal transformations. The vector equation can be split up into two equations

$$L(S)\theta_x - g(S)\frac{\partial S}{\partial t}\,\theta_y\,\frac{eh}{cm} = 0$$

$$L(S)\theta_y + g(S)\,\frac{\partial S}{\partial t}\,\theta_x\frac{eh}{cm} = 0.$$

So that remembering $\theta_x^2+\theta_y^2 = 0$

we have
$$L^2(S)+g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2 = 0$$

which breaks up into two equations

$$g(S)\,p(S)+\frac{Ne^2}{m}(\text{grad } S)^2\pm\iota g(S)\frac{\partial S}{\partial t}\,\frac{eh}{cm} = 0$$

or
$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\text{grad}^2 S+\frac{\dfrac{Ne^2}{mc^2}\left(\dfrac{\partial S}{\partial t}\right)^2}{\dfrac{\partial S}{\partial t}\left(\dfrac{\partial S}{\partial t}+\nu\pm\iota\,\dfrac{eh}{cm}\right)} = 0$$

which can again be put in the form

$$\frac{\mu^2}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-(\text{grad } S)^2 = 0.$$

If
$$\mu^2 = 1+\frac{Ne^2/m}{\dfrac{\partial S}{\partial t}\left(\dfrac{\partial S}{\partial t}+\nu\pm\iota\,\dfrac{eh}{cm}\right)};$$

total reflection is obtained as before by putting $\mu = 0$

$\text{grad}^2\, S$ can also become infinite, if $\dfrac{\partial S}{\partial t} = \pm\dfrac{\iota eh}{cm}-\nu$

i.e., for a frequency equal to the gyromagnetic frequency.

Case III.

Neither $L(S)$ nor $\theta^2 = 0$

i.e., when $(\text{grad } S\theta) \neq 0$

and θ is not parallel to E.

We have to proceed to further elimination to get the refractive index surface.

The elimination work becomes simplified if we choose a new set of axis, with Z axis, in the direction of h.

The vector equation is now equivalent to the three scalar equations

$$L(S)\theta_x - g(S)\frac{\partial S}{\partial t}\theta_y\frac{eh}{cm} = \frac{Ne^2}{m}S_x(S_x\theta_x+S_y\theta_y+S_z\theta_z) \qquad \ldots \quad (i)$$

$$L(S)\theta_y + g(S)\frac{\partial S}{\partial t}\theta_x\frac{eh}{cm} = \frac{Ne^2}{m}S_y(S_x\theta_x+S_y\theta_y+S_z\theta_z) \qquad \ldots \quad (ii)$$

$$L(S)\theta_z = \frac{Ne^2}{m}S_z(S_x\theta_x+S_y\theta_y+S_z\theta_z) \qquad \ldots \quad (iii)$$

where
$$\left(S_x = \frac{\partial S}{\partial x},\ S_y = \frac{\partial S}{\partial y},\ S_z = \frac{\partial S}{\partial z}\right).$$

From (*i*) and (*ii*) we easily get the relations

$$\theta_x\left[L^2(S)+g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\frac{e^2h^2}{c^2m^2}\right]$$

$$= \frac{Ne^2}{m}\left[L(S)S_x + S_y g(S)\frac{\partial S}{\partial t}\ \frac{eh}{cm}\right](S_y\theta_x+S_y\theta_y+S_z\theta_z)$$

$$\theta_y\left[L^2(S)+g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\frac{e^2h^2}{c^2m^2}\right]$$

$$= \frac{Ne^2}{m}\left[L(S)S_y - S_x g(S)\frac{\partial S}{\partial t}\ \frac{eh}{cm}\right][S_x\theta_x+S_y\theta_y+S_z\theta_z]$$

and
$$L_z\theta_z = \frac{Ne^2}{m}S_z[S_x\theta_x+S_y\theta_y+S_z\theta_z].$$

$$\frac{\theta_x}{S_x\theta_x+S_y\theta_y+S_z\theta_z} = \frac{L(S)S_x+S_y g(S)\dfrac{\partial S}{\partial t}\dfrac{eh}{cm}}{L^2(S)+g^2(S)\left(\dfrac{\partial S}{\partial t}\right)^2\dfrac{e^2h^2}{c^2m^2}}\cdot\frac{Ne^2}{m} \qquad \ldots \quad (iv)$$

$$\frac{\theta_y}{S_x\theta_x+S_y\theta_y+S_z\theta_z} = \frac{L(S)S_y-S_x g(S)\dfrac{\partial S}{\partial t}\dfrac{eh}{cm}}{L^2(S)+g^2(S)\left(\dfrac{\partial S}{\partial t}\right)^2\dfrac{e^2h^2}{c^2m^2}}\cdot\frac{Ne^2}{m} \qquad \ldots \quad (v)$$

$$\frac{\theta_z}{S_x\theta_x+S_y\theta_y+S_z\theta_z} = \frac{\dfrac{Ne^2}{m}S_z}{L(S)}; \qquad \ldots \quad (vi)$$

multiplying (*iv*), (*v*) and (*vi*) by S_x, S_y, S_z, we have

$$1 = \frac{Ne^2}{m}\left[\frac{L(S)(S_x^2+S_y^2)}{L^2(S)+g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2}+\frac{S_z^2}{L(S)}\right]$$

or $$L(S)\left[L^2(S)+g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2\right]$$

$$= \frac{Ne^2}{m}\left[L^2(S)(S_x^2+S_y^2+S_z^2)+S_z^2g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2\right].$$

Now writing $$S_z^2 = \text{grad}^2 S\cos^2\alpha$$

where α = the angle between the wavenormal and the magnetic field, we have

$$L^2(S)\left[L(S)-\frac{Ne^2}{m}\text{grad}^2 S\right]+g^2(S)\,L(S)\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2$$

$$=\frac{Ne^2}{m}\left(\frac{eh}{cm}\right)^2 g^2(S)\left(\frac{\partial S}{\partial t}\right)^2\text{grad}^2 S\cos^2\alpha.$$

Remembering $$L(S)-\frac{Ne^2}{m}\text{grad}^2 S = p(S)g(S)$$

and rejecting the factor $g(S)$ the equation finally assumes the form

$$p(S)L^2(S)+g(S)\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2 L(S)$$

$$-\frac{Ne^2}{m}\left(\frac{\partial S}{\partial t}\right)^2 g(S)\left(\frac{eh}{cm}\right)^2\text{grad}^2 S.\cos^2\alpha = 0.$$

When this equation is regarded as a quadratic in ($\text{grad}^2 S$) *i.e.*, when the value of $\frac{\partial S}{\partial t}$ is supposed to be prescribed, the two roots of grad²S, become, let us say, $f_1\left(\frac{\partial S}{\partial t},\alpha\right)$ and $f_2\left\{\frac{\partial S}{\partial t},\alpha\right\}$. So that we get two rays corresponding to any prescribed α.

Corresponding to the two refractive indices

$$\mu_1^2 = \frac{f_1\left\{\frac{\partial S}{\partial t},\alpha\right\}}{\frac{1}{c^2}\left\{\frac{\partial S}{\partial t}\right\}^2},\ \text{and}\ \mu_2^2 = \frac{f_2\left\{\frac{\partial S}{\partial t},\alpha\right\}}{\frac{1}{c^2}\left\{\frac{\partial S}{\partial t}\right\}^2}$$

When damping is neglected, $\frac{\partial S}{\partial t} = \iota\,\omega$.

We see from (*iv*), (*v*) and (*vi*), $\theta_x : \theta_y : \theta_z$

defines a complex-ratio. As $L(S)$ will become real, and $\frac{\partial S}{\partial t}$ pure imaginary, the two split rays will there be elliptically polarised, and the ratio of the axis, projected on a plane perpendicular of the magnetic field, is easily seen to be

$$\frac{g(S)\frac{\partial S}{\partial t}\frac{eh}{cm}}{L(S)}=\frac{g(S)\,\frac{\partial S}{\partial t}\left(\frac{eh}{cm}\right)}{\left[\frac{p(S)}{c^2}-\mathrm{grad}^2S\right]}=\frac{\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\mathrm{grad}^2S\right]\frac{eh}{cm}}{\frac{\partial S}{\partial t}\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\mathrm{grad}^2S+\frac{Ne^2}{mc^2}\left(\frac{\partial S}{\partial t}\right)^2\right]};$$

for large values of grad^2S this tends to $\rightarrow\frac{eh}{cm\,\omega_c\iota}$.

Writing out the equation in extenso and putting $\mathrm{grad}^2\,S=y$, we see the equation

$$0=p(S)\left[\frac{p(S)}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)y\right]^2+\left[\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-y\right]\left(\frac{eh}{cm}\right)^2\left(\frac{\partial S}{\partial t}\right)^2$$

$$\left\{\frac{p(S)}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-y\left(\frac{\partial S}{\partial t}+\nu\right)\frac{\partial S}{\partial t}+\frac{Ne^2}{m}\cos^2\alpha\right]\right\}.$$

The co-efficient of $y^2=p(S)\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)^2+\left(\frac{eh}{cm}\right)^2\left(\frac{\partial S}{\partial t}\right)^2\left[\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)\right.$

$$\left.+\frac{Ne^2}{m}\cos^2\alpha\right].$$

The absolute term on the other hand is equal to

$$\frac{p^3(S)}{c^4}\left(\frac{\partial S}{\partial t}\right)^4+\frac{1}{c^4}\left(\frac{\partial S}{\partial t}\right)^6p(S)\left(\frac{eh}{cm}\right)^2=\frac{p(S)}{c^4}\left(\frac{\partial S}{\partial t}\right)^4\left[p^2(S)+\left(\frac{\partial S}{\partial t}\right)^2\left(\frac{eh}{cm}\right)^2\right].$$

We get therefore

$$\frac{\partial S}{\partial t}\left(\frac{\partial S}{\partial t}+\nu\right)+\frac{\frac{Ne^2}{m}\left[\left(\frac{\partial S}{\partial t}+\nu\right)^2+\left(\frac{eh}{cm}\right)^2\cos^2\alpha\right]}{\left[\left(\frac{\partial S}{\partial t}+\nu\right)^2+\left(\frac{eh}{cm}\right)^2\right]}=0$$

as defining the critical frequencies for total reflection. This is in the general case when collision frequency is not neglected an equation of the fourth degree having two different conjugate complex roots corresponding to two elliptically polarised rays: or when collision frequency is neglected,

and $$\frac{\partial S}{\partial t}=\iota\omega,$$

the above relation reduces to

$$\omega_c^2 = \frac{\dfrac{Ne^2}{m}\left[\omega^2 - \left(\dfrac{eh}{cm}\right)^2 \cos^2\alpha\right]}{\omega^2 - \left(\dfrac{eh}{cm}\right)^2}$$

which agrees with the expression arrived at by Rai, Bajpai, and Mathur, and also with the result arrived at by another method.

From energy consideration, we have seen that when $E \parallel \theta$ the condition of total reflection grad $S.[E \times H) = 0$, is satisfied (grad $S)^2 = 0$: Here the propagation stops from penetrating further, as H vanishes, and the field beyond becomes electrostatic in character. When (grad S $E) \neq 0$, which will happen when the direction of propagation makes an angle with the magnetic field, the electric vector is not the wave front, and the total-reflection condition results when electric vector coincides with the wave normal. This happens as (grad $S)^2 \rightarrow \alpha$. The Poynting vector then lies in the wave front. The flow of energy across the surface stops, and the waves get totally reflected.

CONCLUSION

We have discussed the question of ionosphere at some length, but it will be easily seen, that the method can be applied with the same ease for discussion of propagation of light in a material medium. We have only to utilise the corresponding Lorentz-equation for motion of bound-electrons, which can be made to yield a relation similar to the equation (Z) deduced for the ionosphere. Readers interested in mathematical analysis will easily recognise the method as an application of the method of characteristics, used for discussions of wave-propagation by Hadamard, Debye, and others. So far as the writer is aware the method has not been applied to the microscopic equations of Lorentz, where however the extension does not seem to create any difficulty when the waves are looked upon as possible modes of oscillation of the stationary electro-magnetic field, generated by positive and negative charges, present in the medium.

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STUDIES IN LORENTZ GROUP

BY

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1. The decomposition of any Lorentz transformation into two commutable factors is a well-known result, and forms a convenient starting point for the theory of spinors. Einstein arrived at the decomposition from consideration of the infinitesimal elements of the Lorentz group. The result is here obtained from purely algebraic consideration, which gives at the same time a slightly more general result inasmuch as the factorisation can be shown to hold for elements of the general orthogonal group C_4, with det $C_4 = +1$. The Lorentz group L is really a sub-group of C_4 inasmuch as the co-efficients are restricted by certain reality conditions, for every matrix C in L has C_{rs} real ($r, s = 1, 2, 3$), while C_{4r}, C_{r4}, $r = 1, 2, 3$ are pure imaginaries, and C_{44} real.

If C is a matrix, the transposed matrix will be denoted by $\bar{C}$; a symmetric matrix for which $\bar{C} = C$ will be denoted by S. If $\bar{C}+C = 0$, the matrix is anti-symmetric and will be denoted by A.

A non-symmetric matrix $C = S+A$ while $\bar{C} = S-A$. We shall assume here the elements of a matrix to be complex numbers.

2. Prop. I. An anti-symmetric matrix A having six non-zero elements can be further decomposed into two component matrices P and Q, which have only three non-zero elements such that

$$\left.\begin{array}{lll} P_{12} = P_{34}, & P_{23} = P_{14}, & P_{31} = P_{24} \\ Q_{12} = -Q_{34}, & Q_{23} = -Q_{14}, & Q_{31} = -Q_{24} \end{array}\right\}. \qquad \dots (1)$$

If

$$A = P+Q,$$

$$A_{rs} = P_{rs}+Q_{rs}$$

where

$$\left.\begin{array}{lll} P_{12} = \tfrac{1}{2}(A_{12}+A_{34}), & P_{23} = \tfrac{1}{2}(A_{23}+A_{14}), & P_{31} = \tfrac{1}{2}(A_{31}+A_{24}) \\ Q_{12} = \tfrac{1}{2}(A_{12}-A_{34}), & Q_{23} = \tfrac{1}{2}(A_{23}-A_{14}), & Q_{31} = \tfrac{1}{2}(A_{31}-A_{24}) \end{array}\right\}. \qquad \dots (2)$$

If further we introduce certain basic matrices, defined by

$$P_1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad P_3 = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

and

$$Q_1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad Q_2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad Q_3 = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix},$$

in which the first set P have the properties of the above P matrices in (1), and the second set Q those of the Q matrices. We note that

$$\begin{gathered} P_1^2 = P_2^2 = P_3^2 = Q_1^2 = Q_2^2 = Q_3^2 = -E \\ P_1 = P_2P_3 = -P_3P_2, \quad P_2 = P_3P_1 = -P_1P_3, \quad P_3 = P_1P_2 = -P_2P_1 \\ Q_1 = Q_2Q_3 = -Q_3Q_2, \quad Q_2 = Q_3Q_1 = -Q_1Q_3, \quad Q_3 = Q_1Q_2 = -Q_2Q_1, \\ P_r\, Q_s = Q_s\, P_r. \end{gathered} \qquad \dots \quad (3)$$

Any P matrix can then be represented as

$$P = a_1P_1 + a_2P_2 + a_3P_3, \qquad \dots \quad (4)$$

and any Q matrix as

$$Q = b_1Q_1 + b_2Q_2 + b_3Q_3. \qquad \dots \quad (5)$$

Hence

$$A = a_1P_1 + a_2P_2 + a_3P_3 + b_1Q_1 + b_2Q_2 + b_3Q_3 \qquad \dots \quad (6)$$

Carrying out the multiplication of all P_i's and Q_i's ($i = 1, 2, 3$), we get in all sixteen matrices which we can utilise as basic matrices for the representation of any arbitrary matrix C thus (the indices of Q will be raised in what follows when Q's appear with summations)

$$C = C_0^0 E + \Sigma C_0^r\, P_r + \Sigma C_r^0 Q^r + \Sigma C_\mu^\lambda\, P_\lambda Q^\mu \qquad \dots \quad (7)$$

$$\text{spur of } C = 4C_0^0.$$

Any symmetric matrix

$$S = S_0E + \sum_{r,\, s} S_s^r P_r Q^s. \qquad \dots \quad (7')$$

Since $P_i^2 = -E$, we have generally

$$P^2 = -(a_1^2 + a_2^2 + a_3^2)E. \qquad \dots \quad (8)$$

Prop. II. If S is any orthogonal matrix, *i.e.*, $S^2 = E$, det $S \neq 0$, then, either (*i*) det $S = -1$, spur $S \neq 0$, or (*ii*) det $S = +1$, spur $S = 0$.

We take a symmetric matrix

$$S = \begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & d \end{vmatrix}$$

Since $S^2 = E$, $S = S^{-1}$, and as det $S \neq 0$, the inverse can be written down by utilising the minors of the det S from which the following results follow easily.

If $s =$ det S, and A, B, C, etc. are minors of the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

then

$$\begin{array}{llll} s(ad-x^2) = A & s(fd-yz) = F & BC-F^2 = aD \\ s(bd-y^2) = B & s(gd-zx) = G & \text{etc.} & \text{etc.} \\ s(cd-z^2) = C & s(hd-xy) = H & ,, & ,, \end{array}$$

We also remember the usual orthogonality relations of the type

$$ax+hy+gz+xd = 0, \quad \text{etc.}$$
$$a^2+h^2+g^2+x^2 = 1, \quad \text{etc.}$$

So we easily deduce

$$Ax+Hy+Gz+sx = 0$$
$$Hx+By+Fz+sy = 0$$
$$Gx+Fy+Cz+sz = 0,$$

so that, if x, y, z be not all zero, we have by elimination

$$s^3+s^2(A+B+C)+sD(a+b+c)+D^2 = 0,$$

or, as $s^2 \neq 0$, and $D = sd$

$$(s+A+B+C) = -d(a+b+c+d).$$

Again
$$A+B+C = s[d(a+b+c)-x^2-y^2-z^2]$$
$$= s[d(a+b+c+d)-1];$$

so that
$$(A+B+C+s) = sd(a+b+c+d).$$

Hence either $s = -1$, or $a+b+c+d = 0$, or $d = 0$.

The special cases

$$d = 0 \quad \text{and} \quad x = y = z = 0$$

can be treated in the same way, and we will have in every case $s = -1$ or spur $S = 0$, which proves our general result.

Prop. III. If $S^2 = E$ and spur $S = 0$ (so that $s = +1$), then S is always decomposable into two commutative anti-symmetric factors.

As spur $S = 0$, we can write by (7)

$$S = \Sigma S^\lambda_\mu P_\lambda Q^\mu.$$

Now putting

$$S^\lambda_\mu P_\lambda = \Delta_\mu \qquad \dots \quad (9)$$

where Δ_μ are certain matrices of the P class, we have

$$S = \Delta_1 Q_1 + \Delta_2 Q_2 + \Delta_3 Q_3. \qquad \dots \quad (10)$$

Again as

$$S^2 = E = \sum_r \Delta_r^2 + Q_1(\Delta_2\Delta_3 - \Delta_3\Delta_2) + Q_2(\Delta_3\Delta_1 - \Delta_1\Delta_3) + Q_3(\Delta_1\Delta_2 - \Delta_2\Delta_1),$$

we must have

$$\begin{aligned} \Delta_2\Delta_3 &= \Delta_3\Delta_2 \\ \Delta_3\Delta_1 &= \Delta_1\Delta_3 \\ \Delta_1\Delta_2 &= \Delta_2\Delta_1. \end{aligned} \qquad \dots \quad (11)$$

Hence the Δ matrices are *commutable.*

It follows easily, as the Δ's are all of the same P class,

$$\Delta_1 : \Delta_2 : \Delta_3 = k_1 : k_2 : k_3$$

where k_1, k_2, k_3 are certain numbers:

hence

$$\frac{\Delta_1}{k_1} = \frac{\Delta_2}{k_2} = \frac{\Delta_3}{k_3} = (l_1 P_1 + l_2 P_2 + l_3 P_3) \qquad \dots \quad (12)$$

so that

$$S = (l_1 P_1 + l_2 P_2 + l_2 P_3)(k_1 Q_1 + k_2 Q_2 + k_3 Q_3) \qquad \dots \quad (13)$$

which puts the symmetric orthogonal matrix with zero spur into two commutative anti-symmetric factors.

Also

$$\det S = |\det(l_1 P_1 + l_2 P_2 + l_3 P_3)| \; |\det(k_1 Q_1 + k_2 Q_2 + k_3 Q_3)|$$

whence

$$1 = (l_1^2+l_2^2+l_3^2)^2(k_1^2+k_2^2+k_3^2)^2. \quad \ldots \quad (14)$$

When the elements of S are given, we can determine the elements of the factor matrices in the following way.

As

$$S = \begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & d \end{vmatrix}, \text{ and } \quad a+b+c+d = 0,$$

we can put

$$S = \Delta_1 Q_1+\Delta_2 Q_2+\Delta_3 Q_3.$$

Then

$$\left.\begin{aligned} SQ_1 &= -\Delta_1+(\Delta_3 Q_2-\Delta_2 Q_3) \\ SQ_2 &= -\Delta_2+(\Delta_1 Q_3-\Delta_3 Q_1), \text{ etc.} \end{aligned}\right\} \quad \ldots \quad (15)$$

Hence Δ_1, Δ_2, Δ_3 are obtained by separating the anti-symmetric parts of SQ_1, SQ_2 and SQ_3. These are expressible as follows :

$$\begin{aligned} \Delta_1 &= \tfrac{1}{2}[(a+d)P_1+(h-z)P_2+(g+y)P_3] \\ \Delta_2 &= \tfrac{1}{2}[(h+z)P_1+(b+d)P_2+(f-x)P_3] \\ \Delta_3 &= \tfrac{1}{2}[(g-y)P_1+(f+x)P_2+(c+d)P_3]. \end{aligned} \quad \ldots \quad (16)$$

as

$$\frac{a+d}{2k_1} = \frac{h+z}{2k_2} = \frac{g-y}{2k_3} = l_1$$

$$\frac{h-z}{2k_1} = \frac{b+d}{2k_2} = \frac{f+x}{2k_3} = l_2$$

$$\frac{g+y}{2k_1} = \frac{f-x}{2k_2} = \frac{c+d}{2k_3} = l_3,$$

hence

$$a+d = 2k_1 l_1, \quad b+d = 2k_2 l_2, \quad c+d = 2k_3 l_3.$$

Also

$$a+b+c+d = 0,$$

so

$$\begin{aligned} d &= k_1 l_1+k_2 l_2+k_3 l_3 \\ a &= k_1 l_1-k_2 l_2-k_3 l_3, \text{ etc.} \\ h &= k_1 l_2+k_2 l_1, \\ z &= k_2 l_1-k_1 l_2. \end{aligned}$$

Hence,

$$S = \left| \begin{array}{llll} k_1l_1-k_2l_2-k_3l_3, & k_1l_2+k_2l_1, & k_1l_3+k_3l_1, & k_3l_2-k_2l_3 \\ \text{—} \quad , & k_2l_2-k_1l_1-k_3l_3, & k_2l_3+k_3l_2, & k_1l_3-k_3l_1 \\ \text{—} & \text{—} \quad , & k_3l_3-k_1l_1-k_2l_2, & k_2l_1-k_1l_2 \\ \text{—} & \text{—} & \text{—} \quad , & k_1l_1+k_2l_2+k_3l_3 \end{array} \right.$$

$$= (k_1P_1+k_2P_2+k_3P_3)(l_1Q_1+l_2Q_2+l_3Q_3)\,; \qquad \dots \quad (17)$$

where also

$$k_1^2+k_2^2+k_3^2 = +1,$$

$$l_1^2+l_2^2+l_3^2 = +1.$$

3. *Decomposition of Non-symmetric Matrices*

A general orthogonal matrix can be written as

$$C = S+A \qquad \dots \quad (18)$$

$$C\overline{C} = \overline{C}C = E.$$

where

From this follows

$$SA-AS = 0, \quad S^2-A^2 = E. \qquad \dots \quad (19)$$

Decomposing

$$A = P+Q,$$

where P and Q are linear combinations of P and Q type matrices, we have

$$S^2-P^2-Q^2-2PQ = E$$

i.e.,

$$S^2 = (1+P^2+Q^2)E+2PQ. \qquad \dots \quad (20)$$

Now

$$\text{spur } C = \text{spur } S = \mu \text{ (say)},$$

hence by (7),

$$S = \mu E+\lambda PQ.$$

Again as

$$S^2 = \mu^2+\lambda^2P^2Q^2+2\mu\lambda PQ. \qquad \dots \quad (21)$$

we have comparing (20) and (21), firstly

$$\lambda = \frac{1}{\mu}. \qquad \dots \quad (22)$$

Thus

$$C = \mu E+P+Q+\frac{PQ}{\mu},$$

or
$$\mu C = [\mu+Q][\mu+P]. \qquad \dots \quad (23)$$

Hence if

$$\mu = k_1k_2$$
$$\mu C = [k_1k_2+Q][k_1k_2+P]$$
$$= k_1k_2\left[k_2+\frac{Q}{k_1}\right]\left[k_1+\frac{P}{k_2}\right].$$

Identifying (20) with (21) we put

$$k_2^2-\frac{Q^2}{k_1^2} = 1, \quad k_1^2-\frac{P^2}{k_2^2} = 1,$$
$$\mu^2-Q^2 = k_1^2, \quad \mu^2-P^2 = k_2^2.$$

Hence

$$(\mu^2-Q^2)(\mu^2-P^2) = \mu^2$$

which relation is equivalent to $\mu^2+\frac{Q^2P^2}{\mu^2} = 1+P^2+Q^2.$

The numbers k_1 and k_2 are thus determined. Thus (23) which is the factorisation aimed at is completely established.

The general form of C can also be calculated.

If

$$C = AB$$

where

$$\left.\begin{aligned} A &= A_0+A_1P_1+A_2P_2+A_3P_3 \\ B &= B_0+B_1Q_1+B_2Q_2+B_3Q_3 \end{aligned}\right\} \qquad \dots \quad (24)$$

and

$$A_0^2+A_1^2+A_2^2+A_3^2 = 1$$
$$B_0^2+B_1^2+B_2^2+B_3^2 = 1.$$

Explicitly,

$$A = \begin{vmatrix} A_0 & -A_3 & A_2 & -A_1 \\ A_3 & A_0 & -A_1 & -A_2 \\ -A_2 & A_1 & A_0 & -A_3 \\ A_1 & A_2 & A_3 & A_0 \end{vmatrix} \quad B = \begin{vmatrix} B_0 & -B_3 & B_2 & +B_1 \\ B_3 & B_0 & -B_1 & +B_2 \\ -B_2 & B_1 & B_0 & +B_3 \\ -B_1 & -B_2 & -B_3 & B_0 \end{vmatrix} \qquad \dots \quad (25)$$

Multiplying we have

$$C = \begin{vmatrix} (A_0B_0+A_1B_1) & -(A_3B_0+A_0B_3) & +(A_2B_0+A_0B_2) & -(A_1B_0-A_0B_1) \\ -(A_2B_2+A_3B_3) & +(A_1B_2+A_2B_1) & +(A_1B_3+A_3B_1) & +(A_2B_3-A_3B_2) \\[1ex] (A_3B_0+A_0B_3) & (A_0B_0+A_2B_2) & -(A_0B_2+A_1B_0) & -(A_2B_0-A_0B_2) \\ +(A_1B_2+A_2B_1) & -(A_3B_3+A_1B_1) & +(A_2B_3+A_3B_2) & +(A_3B_1-A_1B_3) \\[1ex] -(A_2B_0+A_0B_2) & +(A_1B_0+A_0B_1) & (A_0B_0+A_3B_3) & -(A_3B_0-A_0B_3) \\ +(A_1B_3+A_3B_1) & (A_2B_3+A_3B_2) & -(A_1B_1+A_2B_2) & +(A_1B_2-A_2B_1) \\[1ex] (A_1B_0-A_0B_1) & +(A_2B_0-A_0B_2) & (A_3B_0-A_0B_3) & +(A_1B_1+A_2B_2) \\ +(A_2B_3-A_3B_2) & +(A_3B_1-A_1B_3) & +(A_1B_2-A_2B_1) & +(A_3B_3+A_0B_0) \end{vmatrix} \quad \ldots \quad (26)$$

Now

$$\det C = (A_0^2+A_1^2+A_2^2+A_3^2)^2(B_0^2+B_1^2+B_2^2+B_3^2)^2 = 1$$

and

$$\text{spur } C = 4A_0B_0. \qquad \ldots \quad (27)$$

Two special forms of C can be at once noticed :

If C is a matric of the proper Lorentz group, then

$$C_{14},\ C_{24},\ C_{34},\ C_{41},\ C_{42},\ C_{43} \text{ are pure imaginaries,}$$

so that

$$(A_2B_3-A_3B_2)+(A_1B_0-B_1A_0) = i\lambda,$$

$$(A_2B_3-A_3B_2)-(A_1B_0-B_1A_0) = i\lambda',$$

from which follows

$$(A_2B_3-A_3B_2) \text{ as well as } (A_1B_0-B_1A_0),$$

etc, are all pure imaginaries.

A_0B_0, A_1B_1, A_2B_2, A_3B_3 are real quantities as well as $(A_2B_3+A_3B_2)$, etc. and $(A_3B_0+A_0B_3)$, etc. are real.

We have then

$$\frac{A_0}{B_0^*} = \frac{A_1}{B_1^*} = \frac{A_2}{B_2^*} = \frac{A_3}{B_3^*} = k, \text{ real} \qquad \ldots \quad (28)$$

as $A_0^2+A_1^2+A_2^2+A_3^2 = B_0^2+B_1^2+B_2^2+B_3^2 = +1$ (B_0^* representing the conjugate of B_0, etc.).

We thus see that the sets (A_0, A_1, A_2, A_3) and (B_0, B_1, B_2, B_3) *are conjugate to each other.*

There are now two cases possible.

Case I. When (A_0, A_1, A_2, A_3) are real, then

$$A_0 = B_0, \quad A_1 = B_1, \quad A_2 = B_2, \quad A_3 = B_3$$

and from (26) further

$$C_{14} = C_{24} = C_{34} = C_{41} = C_{42} = C_{43} = 0$$
$$C_{44} = 1.$$

This represents a three dimensional rotation.

The axis of rotation is easily seen to be

$$\frac{l}{A_1} = \frac{m}{A_2} = \frac{n}{A_3} \qquad \dots \quad (29)$$

The angle of rotation is given by

$$\tan \psi = \frac{2A_0G}{A_0^2 - G^2}, \text{ where } G^2 = A_1^2 + A_2^2 + A_3^2.$$

Case II. A_1, A_2, A_3 are pure imaginaries, A_0 real.

The resultant transformation is here seen to be an Einstein transformation, with translation along the line

$$\frac{l}{A_1} = \frac{m}{A_2} = \frac{n}{A_3}$$

and velocity given by

$$\frac{V}{c} = \frac{2A_0G}{A_0^2 + G^2}$$

where

$$A_0^2 - G^2 = 1, \quad \text{and} \quad G^2 = A_1^2 + A_2^2 + A_3^2.$$

4.

We shall now consider certain general matrices of the form

$$\left.\begin{aligned} A &= \mu E + aP_1 + bP_2 + cP_3 \\ \text{and} \qquad B &= \mu E + aQ_1 + bQ_2 + cQ_3. \end{aligned}\right\} \qquad \dots \quad (30)$$

We can easily see that

$$|\det A| = [\mu^2 + a^2 + b^2 + c^2]^2$$
$$|\det B| = [\mu^2 + a^2 + b^2 + c^2]^2.$$

So when a, b, c are complex quantities and $|\det A|$ and $|\det B|$ real and positive we must have

$$\mu^2+a^2+b^2+c^2 = \mu^{*2}+a^{*2}+b^{*2}+c^{*2}$$

where μ^*, etc. are conjugates of μ, etc.

We now consider A and B to be factors of decomposition of C, and take the determinant of each of the factors to be $+1$, so that

$$\mu^2+a^2+b^2+c^2 = \pm 1.$$

The matrices of A and B types are orthogonal matrices; but since their elements are generally complex, they do not belong to the Lorentz sub-group of Group C_4.

We will now discuss certain general characteristics of the A and B matrices, which will bring into light also the intimate relation that exists between the decompositions as carried out here and the spinor theory.

Prop. All matrices of A and B types are reducible :

$$A = \left|\begin{array}{cc|cc} p_0 & -p_3 & p_2 & -p_1 \\ p_3 & p_0 & -p_1 & -p_2 \\ \hline -p_2 & p_1 & p_0 & -p_3 \\ p_1 & p_2 & p_3 & p_0 \end{array}\right| = \left|\begin{array}{cc} S & -Q \\ Q & S \end{array}\right| \qquad \dots \quad (31)$$

where S and Q are certain matrices of the second order. That such matrices are reducible is a well-known result.

Let

$$\psi = A\phi$$

define a transformation of a four-component ϕ to a four component ψ.

We have

$$\begin{aligned} \psi_1 &= p_0\phi_1-p_3\phi_2+p_2\phi_3-p_1\phi_4 \\ \psi_2 &= p_3\phi_1+p_0\phi_2-p_1\phi_3-p_2\phi_4 \\ \psi_3 &= -p_2\phi_1+p_1\phi_2+p_0\phi_3-p_3\phi_4 \\ \psi_4 &= p_1\phi_1+p_2\phi_2+p_3\phi_3+p_0\phi_4. \end{aligned} \qquad \dots \quad (32)$$

From this it can be easily deduced that

$$\left|\begin{array}{c} \psi_1+i\psi_2 \\ \psi_3-i\psi_4 \end{array}\right| = \left|\begin{array}{cc} p_0+ip_3 & p_2-ip_1 \\ -p_2-ip_1 & p_0-ip_3 \end{array}\right| \left|\begin{array}{c} \phi_1+i\phi_2 \\ \phi_3-i\phi_4 \end{array}\right| \qquad \dots \quad (33)$$

and there is also the corresponding conjugate relation obtained by putting $-i$ for i, (33) can also be written as

$$\begin{vmatrix} \psi_3+i\psi_4 \\ -\psi_1+i\psi_2 \end{vmatrix} = \begin{vmatrix} p_0+ip_3 & p_2-ip_1 \\ -p_2-ip_1 & p_0-ip_3 \end{vmatrix} \begin{vmatrix} \phi_3+i\phi_4 \\ i\phi_2-\phi_1 \end{vmatrix} \quad \ldots \quad (34)$$

Combining (33) and (34) we obtain

$$\begin{vmatrix} \psi_3+i\psi_4 & \psi_1+i\psi_2 \\ i\psi_2-\psi_1 & \psi_3-i\psi_4 \end{vmatrix} = \begin{vmatrix} p_0+ip_3 & p_2-ip_1 \\ -p_2-ip_1 & p_0-ip_3 \end{vmatrix} \begin{vmatrix} \phi_3+i\phi_4 & \phi_1+i\phi_2 \\ i\phi_2-\phi_1 & \phi_3-i\phi_4 \end{vmatrix} \quad \ldots \quad (35)$$

which is a well-known spinor transformation.

Again, choosing two sets of variables as (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) the corresponding transformation formula for B matrices can be written as

$$\begin{vmatrix} y_3+iy_4 & -y_1+iy_2 \\ y_1+iy_2 & y_3-iy_4 \end{vmatrix} = \begin{vmatrix} q_0-iq_3 & -q_2-iq_1 \\ q_2-iq_1 & q_0+iq_3 \end{vmatrix} \begin{vmatrix} x_3+ix_4 & -x_1+ix_2 \\ x_1+ix_2 & x_3-ix_4 \end{vmatrix} \quad \ldots \quad (36)$$

which can be also written in the form

$$\begin{vmatrix} y_3+iy_4 & y_1+iy_2 \\ -y_1+iy_2 & y_3-iy_4 \end{vmatrix} = \begin{vmatrix} x_3+ix_4 & x_1+ix_2 \\ -x_1+ix_2 & x_3-ix_4 \end{vmatrix} \begin{vmatrix} q_0-iq_3 & q_2-iq_1 \\ -q_2-iq_1 & q_0+iq_3 \end{vmatrix}. \quad \ldots \quad (37)$$

(35) can also be written in this notation with the transformation matrix to the left on the right-hand side.

Thus A can be regarded as inducing a front transformation while B induces a back transformation, or *vice-versa*, which shows the commutative nature of the transformation at once.

The general reducibility of the A and B types of matrices shows that the Lorentz Group can be studied advantageously in the two-dimensional representation of the A and B class of transformation.

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THE COMPLETE SOLUTION OF THE EQUATION:

$\nabla^2\phi - \frac{\partial^2\phi}{c^2\,\partial t^2} - k^2\phi = -4\pi\rho(xyzt).$

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1. In what follows we desire to present what appears to us to be a complete Kirchhoff-like solution of the equation above and indeed by two distinct methods. The one is a method of complex integration and the other an extension of a method adopted by Love * for ordinary retarded potentials. The equation itself is one which seems to have acquired some importance in view of a recent work of J. H. Bhabha † on the mesotron, who has stated a solution of the equation, in which the part involving a surface integral does not appear. To the best of our knowledge and belief the solution at which we arrive and which exhibits the surface integral is original. As the whole work, however, was started by a successful derivation, by an adaptation of the well-known method of Herglotz and Sommerfeld ‡, of the potentials of a moving mesotron, such as Bhabha uses, we let this derivation precede the main investigation, since this way of arriving at the potentials appears to us to be also new and of sufficient interest.

2. To start with, we seek a solution of the equation

$$\nabla^2\psi - \frac{\partial^2\psi}{c^2\,\partial\theta^2} - k^2\psi = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

of the form: $\psi = \psi(s)$,

where
$$s^2 = c^2(\theta-t)^2 - \Sigma(\xi-x)^2 = c^2(\theta-t)^2 - r^2 \qquad \ldots \quad \ldots \quad (2)$$

and x, y, z, t are parameters. The transformation of eq. (1) to the single variable s leads to the equation

$$\frac{d^2\psi}{ds^2} + \frac{3}{s}\frac{d\psi}{ds} + k^2\psi = 0$$

or
$$\frac{d^2}{ds^2}(s\psi) + \frac{1}{s}\frac{d}{ds}(s\psi) + \left(k^2 - \frac{1}{s^2}\right)(s\psi) = 0. \qquad \ldots \quad \ldots \quad (3)$$

Hence we infer: $\psi = \frac{J_1(ks)}{s}$ or $\frac{Y_1(ks)}{s}$, where $J_1(ks)$ and $Y_1(ks)$ are Bessel functions of the first and second kinds respectively.

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3. *Retarded potentials of a moving mesotron.*

We put $\chi = \dfrac{kY_1(ks)}{s}$

$$= \frac{kJ_1(ks)}{s} ln(ks) - \frac{k}{s} \sum_{m=0}^{\infty} (-1)^m \frac{2m+1}{m(m+1)} J_{2m+1}(ks) - \frac{J_0(ks)}{s^2} \S \qquad \ldots \quad (4)$$

and choose for the four potentials (ϕ, A_1, A_2, A_3) of the mesotron solutions of eq. (1), stated as an equation in x, y, z, t, with ξ, η, ζ, θ as parameters, in the forms

$$\phi = \int_{\theta_1}^{\theta_2} d\theta f_4(\theta)\chi, \quad A_1 = \int_{\theta_1}^{\theta_2} d\theta\, f_1(\theta)\chi, \text{ etc.} \qquad \ldots \quad (5)$$

The equation of restraint on the four potentials becomes

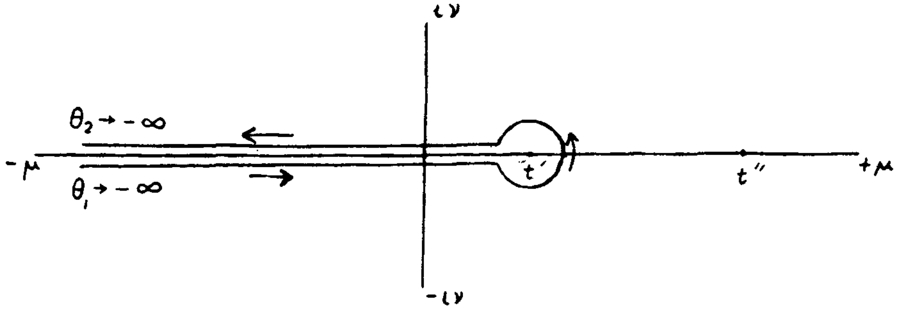
$$\begin{aligned} 0 &= \frac{\partial\phi}{c\,\partial t} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \\ &= -\int_{\theta_1}^{\theta_2} d\theta \left[f_4 \frac{\partial}{c\,\partial\theta} + f_1 \frac{\partial}{\partial\xi} + f_2 \frac{\partial}{\partial\eta} + f_3 \frac{\partial}{\partial\zeta} \right] \chi \\ &= -B \int_{\theta_1}^{\theta_2} d\theta \left[\frac{\partial}{\partial\theta} + \frac{d\xi}{d\theta}\frac{\partial}{\partial\xi} + \frac{d\eta}{d\theta}\frac{\partial}{\partial\eta} + \frac{d\zeta}{d\theta}\frac{\partial}{\partial\zeta} \right] \chi \\ &= -B \int_{\theta_1}^{\theta_2} d\theta \frac{d\chi}{d\theta} \qquad \ldots \quad (6) \end{aligned}$$

if, after Herglotz and Sommerfeld, we treat ξ, η, ζ as functions of θ and put

$$(f_4, f_1, f_2, f_3) = B\left(c, \frac{d\xi}{d\theta}, \frac{d\eta}{d\theta}, \frac{d\zeta}{d\theta}\right), \qquad \ldots \quad (7)$$

where B is an absolute constant.

We assume now θ to have been complex of the form $\mu + i\nu$ and proceed to perform the integrations in eqs. (5) along a path on the complex plane shown in the diagram below.



The following considerations govern the choice of this path. Since, from eq. (2),

$$s^2=c^2(\theta-t')(\theta-t''), \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where $t'=t-\frac{r}{c}$ and $t''=t+\frac{r}{c}$, the only singularities of χ are those at $\theta=t'$ and $\theta=t''$ and since we are seeking, for the moment, the retarded potentials we let the path embrace just the one singularity at $\theta=t'$. Since, further, $ln s$ in χ is multiple-valued we cannot satisfy the equation of constraint (6) by just having a closed contour about t' but have to let the path run along the real axis from $-\infty$ on one Riemann surface round t' to $-\infty$ on a second, as, with $\theta\to-\infty$, $\chi\to 0$.

Retaining only such terms in χ as involve a singularity we may write eqs. (5) as

$$\phi=Bc\int_{\theta_1}^{\theta_2}d\theta\left[\frac{kJ_1(ks)}{s}\,ln s-\frac{1}{s^2}\right],$$

$$A_1=B\int_{\theta_1}^{\theta_2}d\theta\left[\frac{kJ_1(ks)}{s}\,ln s-\frac{1}{s^2}\right]\frac{d\xi}{d\theta},$$

etc.,

and evaluate the integrals as follows:

$$\int_{\theta_1}^{\theta_2}d\theta\frac{1}{s^2}\frac{d\xi}{d\theta}=\oint d\theta\frac{1}{s^2}\frac{d\xi}{d\theta}=\frac{2\pi i}{\left[\frac{d(s^2)}{d\theta}\right]_{t'}}\cdot\frac{dx'}{dt'}=-\frac{\pi i}{cr\left(1-\frac{v_r}{c}\right)}\,v_x, \text{ etc.,}$$

$$\int_{\theta_1}^{\theta_2}d\theta\frac{J_1(ks)}{s}\frac{d\xi}{d\theta}ln s=\frac{1}{2}\int_{\theta_1}^{\theta_2}d\theta\frac{J_1(ks)}{s}\frac{d\xi}{d\theta}ln(\theta-t')$$

$$=-\pi i\int_{-\infty}^{t'}d\theta\frac{J_1(ks)}{s}\frac{d\xi}{d\theta}, \text{ etc.}$$

We identify now $B\pi i$ with the charge e of the mesotron and obtain as its potentials

$$\left.\begin{aligned}\phi&=\frac{e}{r\left(1-\frac{v_r}{c}\right)}-eck\int_{-\infty}^{t'}d\theta\frac{J_1(ks)}{s},\\(A_1,A_2,A_3)&=\frac{e\vec{v}}{cr\left(1-\frac{v_r}{c}\right)}-ek\int_{-\infty}^{t'}d\theta\frac{J_1(ks)}{s}\left(\frac{d\xi}{d\theta},\frac{d\eta}{d\theta},\frac{d\zeta}{d\theta}\right),\end{aligned}\right\}\quad \ldots \quad (9)$$

which are the forms used by Bhabha.

In concluding this section we remark that the advanced potentials of the mesotron, also used by Bhabha, may be obtained in like manner by the choice of a path of integration made to run along the real axis from $+\infty$ on one Riemann surface round the other singularity at $\theta = t''$ back to $+\infty$ on a second.

4. *Solution of the proposed equation for retarded values of t : First method.* We write

$$\nabla^2\phi - \frac{\partial^2\phi}{c^2\,\partial\theta^2} - k^2\phi = -4\pi\rho(\xi\eta\zeta\theta) \qquad \ldots \quad \ldots \quad (10)$$

and

$$\nabla^2\chi - \frac{\partial^2\chi}{c^2\,\partial\theta^2} - k^2\chi = 0,$$

where χ is given by (4). Hence we obtain firstly,

$$\sum \frac{\partial}{\partial\xi}\left(\chi\frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\chi}{\partial\xi}\right) - \frac{\partial}{c\,\partial\theta}\left(\chi\frac{\partial\phi}{c\,\partial\theta} - \phi\frac{\partial\chi}{c\,\partial\theta}\right) = -4\pi\chi\rho(\xi\eta\zeta\theta)$$

and then, upon integration with respect to θ,

$$\int_{\theta_1}^{\theta_2} d\theta \sum \frac{\partial}{\partial\xi}\left(\chi\frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\chi}{\partial\xi}\right) - \frac{1}{c}\left[\chi\frac{\partial\phi}{c\,\partial\theta} - \phi\frac{\partial\chi}{c\,\partial\theta}\right]_{\theta_1}^{\theta_2} = -4\pi\int_{\theta_1}^{\theta_2} d\theta\,\chi\rho(\xi\eta\zeta\theta).$$

We assume now θ to have been complex of the form $\mu + i\nu$ and proceed to perform the integrations along the same path on the complex plane as we have used in the previous section. Since, with $\theta \rightarrow -\infty$, both χ and $\frac{\partial\chi}{\partial\theta} \rightarrow 0$, we have

$$\int_{\theta_1}^{\theta_2} d\theta \sum \frac{\partial}{\partial\xi}\left(\chi\frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\chi}{\partial\xi}\right) = -4\pi\int_{\theta_1}^{\theta_2} d\theta\,\chi\rho(\xi\eta\zeta\theta).$$

This we may write again in the form

$$\sum \frac{\partial}{\partial\xi}\int_{\theta_1}^{\theta_2} d\theta\left(\chi\frac{\partial\phi}{\partial\xi} - \phi\frac{\partial\chi}{\partial\xi}\right) = -4\pi\int_{\theta_1}^{\theta_2} d\theta\,\chi\rho(\xi\eta\zeta\theta), \qquad \ldots \quad (11)$$

if we observe that we may treat the path of integration as independent of ξ, η, ζ, since the path about the singularity at t' may always be slightly varied to accommodate the shift of t' involved in the operations, $\frac{\partial}{\partial\xi}$, etc.

We set down now in brief the necessary calculations.

$$\text{(i)} \quad \int_{\theta_1}^{\theta_2} d\theta\,\chi\rho(\xi\eta\zeta\theta) = \int_{\theta_1}^{\theta_2} d\theta\left[\frac{kJ_1(ks)}{s}\,\ln s - \frac{1}{s^2}\right]\rho(\xi\eta\zeta\theta)$$

$$= -\pi ik\int_{-\infty}^{t'} d\theta\,\frac{J_1(ks)}{s}\,\rho(\xi\eta\zeta\theta) + \frac{\pi i}{cr}\,\rho(\xi\eta\zeta t').$$

$$\text{(ii)}\quad \int_{\theta_1}^{\theta_2} d\theta\,\chi\,\frac{\partial\phi}{\partial\xi} = \int_{\theta_1}^{\theta_2} d\theta\left[\frac{kJ_1(ks)}{s}\,lns - \frac{1}{s^2}\right]\frac{\partial\phi}{\partial\xi}$$

$$= -\pi ik\int_{-\infty}^{t'} d\theta\,\frac{J_1(ks)}{s}\,\frac{\partial\phi}{\partial\xi} + \frac{\pi i}{cr}\left(\frac{\partial\phi}{\partial\xi}\right)_{t'}.$$

$$\text{(iii)}\quad \int_{\theta_1}^{\theta_2} d\theta\,\phi\,\frac{\partial\chi}{\partial\xi} = -(\xi - x)\int_{\theta_1}^{\theta_2} d\theta\,\phi\,\frac{1}{s}\,\frac{d\chi}{ds}$$

$$= -(\xi - x)\int_{\theta_1}^{\theta_2} d\theta\,\phi\,\frac{1}{s}\,\frac{d}{ds}\left[\frac{kJ_1(ks)}{s}\,lns - \frac{J_0(ks)}{s^2}\right]$$

$$= -(\xi - x)\int_{\theta_1}^{\theta_2} d\theta\,\phi\left[k\frac{d}{ds}\left(\frac{J_1(ks)}{s}\right)\frac{lns}{s} + \frac{2kJ_1(ks)}{s^3} + \frac{2J_0(ks)}{s^4}\right]$$

$$= -(\xi - x)\left[k\int_{\theta_1}^{\theta_2} d\theta\,\phi\,\frac{d}{ds}\left(\frac{J_1(ks)}{s}\right)\frac{lns}{s} + \frac{k^2}{2}\int_{\theta_1}^{\theta_2} d\theta\,\frac{\phi}{s^2} + 2\int_{\theta_1}^{\theta_2} d\theta\,\frac{\phi}{s^4}\right]$$

Replacing $\frac{1}{s^4}$ by $\frac{1}{c^4(t''-t')^2}\left[\frac{1}{(\theta - t')^2} - \frac{2}{(\theta - t')(\theta - t'')} + \frac{1}{(\theta - t'')^2}\right]$

we have

$$\int_{\theta_1}^{\theta_2} d\theta\,\phi\,\frac{\partial\chi}{\partial\xi}$$

$$= -(\xi - x)\left[-\pi ik\int_{-\infty}^{t'} d\theta\,\phi\,\frac{d}{ds}\left(\frac{J_1(ks)}{s}\right)\frac{1}{s} - \frac{k^2\pi i}{2cr}(\phi)_{t'} + \frac{\pi i}{cr^2}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t'} + \frac{\pi i}{cr^3}(\phi)_{t'}\right]$$

$$= -\pi ik\int_{-\infty}^{t'} d\theta\,\phi\,\frac{\partial}{\partial\xi}\left(\frac{J_1(ks)}{s}\right) + \frac{k^2\pi i}{2c}(\phi)_{t'}\frac{\partial r}{\partial\xi} - \frac{\pi i}{cr}\,\frac{\partial r}{\partial\xi}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t'} + \frac{\pi i}{c}(\phi)_{t'}\frac{\partial}{\partial\xi}\left(\frac{1}{r}\right).$$

Substituting these results in eq. (11) and dropping the factor $\frac{\pi i}{c}$ we get

$$\begin{aligned}
-4\pi&\left[\frac{\rho(\xi\eta\zeta t')}{r}-ck\int_{-\infty}^{t'}d\theta\,\frac{J_1(ks)}{s}\,\rho(\xi\eta\zeta\theta)\right]\\
&=\sum\frac{\partial}{\partial\xi}\left[\frac{1}{r}\left(\frac{\partial\phi}{\partial\xi}\right)_{t'}+\frac{1}{r}\,\frac{\partial r}{\partial\xi}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t'}-(\phi)_{t'}\frac{\partial}{\partial\xi}\left(\frac{1}{r}\right)-ck\int_{-\infty}^{t'}d\theta\,\frac{J_1(ks)}{s}\,\frac{\partial\phi}{\partial\xi}\right.\\
&\qquad\left.+ck\int_{-\infty}^{t'}d\theta\,\phi\frac{\partial}{\partial\xi}\left(\frac{J_1(ks)}{s}\right)-\frac{k^2}{2}(\phi)_{t'}\frac{\partial r}{\partial\xi}\right]. \quad \ldots \quad (12)
\end{aligned}$$

We integrate this equation now further through a closed volume about the point P $(x,\ y,\ z)$ but exclude the volume of a small sphere about that point. We obtain thus

$$\begin{aligned}
-4\pi&\iiint d\xi\,d\eta\,d\zeta\left[\frac{\rho(\xi\eta\zeta t')}{r}-ck\int_{-\infty}^{t'}d\theta\,\frac{J_1(ks)}{s}\,\rho(\xi\eta\zeta\theta)\right]\\
&=\iint dS\left[\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_{t'}+\frac{1}{r}\,\frac{\partial r}{\partial n}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t'}-(\phi)_{t'}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right.\\
&\qquad\left.-ck\int_{-\infty}^{t'}d\theta\,\frac{J_1(ks)}{s}\,\frac{\partial\phi}{\partial n}+ck\int_{-\infty}^{t'}d\theta\,\phi\frac{\partial}{\partial n}\left(\frac{J_1(ks)}{s}\right)-\frac{k^2}{2}(\phi)_{t'}\frac{\partial r}{\partial n}\right]\\
&\quad+\iint dS'\left[\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_{t'}+\frac{1}{r}\,\frac{\partial r}{\partial n}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t'}-(\phi)_{t'}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right.\\
&\qquad\left.-ck\int_{-\infty}^{t'}d\theta\,\frac{J_1(ks)}{s}\,\frac{\partial\phi}{\partial n}+ck\int_{-\infty}^{t'}d\theta\,\phi\frac{\partial}{\partial n}\left(\frac{J_1(ks)}{s}\right)-\frac{k^2}{2}(\phi)_{t'}\frac{\partial r}{\partial n}\right],
\end{aligned}$$

where the second surface integral is over the surface of the small sphere about P. Its limiting value, with $r\to 0$, is seen to be

$$-\phi_{P,t}\iint dS'\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)=-4\pi\phi_{P,t}.$$

$$\nabla^2\phi - \frac{\partial^2\phi}{c^2\,\partial t^2} - k^2\phi = -4\pi\rho(xyzt).$$

We arrive hence at the complete Kirchhoff-like solution for retarded values of t in the form

$$\phi(xyzt) = \iiint d\xi\, d\eta\, d\zeta \left[\frac{\rho(\xi\eta\zeta t')}{r} - ck \int_{-\infty}^{t'} d\theta\, \frac{J_1(ks)}{s}\, \rho(\xi\eta\zeta\theta) \right]$$

$$+\frac{1}{4\pi}\iint dS \left[\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_{t'} + \frac{1}{r}\frac{\partial r}{\partial n}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t'} - (\phi)_{t'}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right.$$

$$\left. -ck\int_{-\infty}^{t'} d\theta \left\{ \frac{J_1(ks)}{s}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{J_1(ks)}{s}\right) \right\} - \frac{k^2}{2}(\phi)_{t'}\frac{\partial r}{\partial n} \right]. \quad \ldots \quad (13)$$

In closing this section we remark that a like procedure with the second path of integration pointed out in the previous section yields the complete Kirchhoff-like solution for advanced values of t in the form

$$\phi(xyzt) = \iiint d\xi\, d\eta\, d\zeta \left[\frac{\rho(\xi\eta\zeta t'')}{r} - ck \int_{t''}^{\infty} d\theta\, \frac{J_1(ks)}{s}\, \rho(\xi\eta\zeta\theta) \right]$$

$$+\frac{1}{4\pi}\iint dS \left[\frac{1}{r}\left(\frac{\partial\phi}{\partial n}\right)_{t''} - \frac{1}{r}\frac{\partial r}{\partial n}\left(\frac{\partial\phi}{c\,\partial\theta}\right)_{t''} - (\phi)_{t''}\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \right.$$

$$\left. -ck\int_{t''}^{\infty} d\theta \left\{ \frac{J_1(ks)}{s}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{J_1(ks)}{s}\right) \right\} - \frac{k^2}{2}(\phi)_{t''}\frac{\partial r}{\partial n} \right]. \quad \ldots \quad (14)$$

5. *Solution of the proposed equation for retarded values of t : Second method.*

To begin with, we replace θ by $t-\frac{r}{c}\,(=t')$ in eq. (10). With the notation

$$\left.\begin{aligned} \rho\left(\xi,\eta,\zeta,t-\frac{r}{c}\right) &= [\rho], \\ \phi\left(\xi,\eta,\zeta,t-\frac{r}{c}\right) &= [\phi], \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

etc.,

equation (10) takes then the form

$$[\nabla^2\phi] - \left[\frac{\partial^2\phi}{c^2\,\partial\theta^2}\right] - k^2[\phi] = -4\pi[\rho]. \quad \ldots \quad \ldots \quad (16)$$

Now

$$\left.\begin{aligned}
\frac{\partial}{\partial \xi}[\phi] &= \left[\frac{\partial \phi}{\partial \xi}\right]-\left[\frac{\partial \phi}{\partial \theta}\right]\frac{1}{c}\frac{\partial r}{\partial \xi},\\
\frac{\partial}{\partial t}[\phi] &= \left[\frac{\partial \phi}{\partial \theta}\right],\\
\frac{\partial^2}{\partial \xi^2}[\phi] &= \left[\frac{\partial^2 \phi}{\partial \xi^2}\right]-2\left[\frac{\partial^2 \phi}{\partial \xi\, \partial \theta}\right]\frac{1}{c}\frac{\partial r}{\partial \xi}\\
&\quad+\left[\frac{\partial^2 \phi}{\partial \theta^2}\right]\left(\frac{1}{c}\frac{\partial r}{\partial \xi}\right)^2-\left[\frac{\partial \phi}{\partial \theta}\right]\frac{1}{c}\frac{\partial^2 r}{\partial \xi^2},\\
\frac{\partial^2}{\partial \xi\, \partial t}[\phi] &= \left[\frac{\partial^2 \phi}{\partial \xi\, \partial \theta}\right]-\left[\frac{\partial^2 \phi}{\partial \theta^2}\right]\frac{1}{c}\frac{\partial r}{\partial \xi}.
\end{aligned}\right\} \quad \ldots \quad (17)$$

Hence we deduce firstly

$$\frac{\partial^2}{\partial \xi^2}[\phi]+\frac{2}{c}\frac{\partial r}{\partial \xi}\frac{\partial^2}{\partial \xi\, \partial t}[\phi]+\frac{1}{c}\frac{\partial^2 r}{\partial \xi^2}\frac{\partial}{\partial t}[\phi] = \left[\frac{\partial^2 \phi}{\partial \xi^2}\right]-\left(\frac{1}{c}\frac{\partial r}{\partial \xi}\right)^2\left[\frac{\partial^2 \phi}{\partial \theta^2}\right]$$

and then

$$\begin{aligned}
\nabla^2[\phi]+\frac{2}{c}\sum\frac{\partial r}{\partial \xi}\frac{\partial^2}{\partial \xi\, \partial t}[\phi]+\frac{2}{cr}\frac{\partial}{\partial t}[\phi]-k^2[\phi] &= [\nabla^2\phi]-\left[\frac{\partial^2 \phi}{c^2\, \partial \theta^2}\right]-k^2[\phi]\\
&= -4\pi[\rho] \quad \ldots \quad \ldots \quad \ldots \quad (18)
\end{aligned}$$

in view of eq. (16). Dividing both sides of eq. (18) by r, we have, since $\nabla^2\frac{1}{r}=0$,

$$\sum\frac{\partial}{\partial \xi}\left\{\frac{1}{r}\frac{\partial}{\partial \xi}[\phi]-[\phi]\frac{\partial}{\partial \xi}\left(\frac{1}{r}\right)+\frac{2}{c}\frac{1}{r}\frac{\partial r}{\partial \xi}\frac{\partial}{\partial t}[\phi]\right\}-\frac{k^2[\phi]}{r} = -4\pi\frac{[\rho]}{r}. \quad \ldots \quad (A)$$

We next proceed to combine eq. (10) with

$$\nabla^2 Q-\frac{\partial^2 Q}{c^2\, \partial \theta^2}-k^2 Q = 0,$$

where
$$Q = \frac{J_1(ks)}{s},$$

and obtain firstly

$$\sum\frac{\partial}{\partial \xi}\left(Q\frac{\partial \phi}{\partial \xi}-\phi\frac{\partial Q}{\partial \xi}\right)-\frac{\partial}{c\, \partial \theta}\left(Q\frac{\partial \phi}{c\, \partial \theta}-\phi\frac{\partial Q}{c\, \partial \theta}\right) = -4\pi Q\rho(\xi\eta\zeta\theta)$$

and then, since both Q and $\frac{\partial Q}{\partial \theta}\rightarrow 0$ with $\theta\rightarrow-\infty$,

$$\sum\int_{-\infty}^{t'} d\theta\,\frac{\partial}{\partial \xi}\left(Q\frac{\partial \phi}{\partial \xi}-\phi\frac{\partial Q}{\partial \xi}\right)-\frac{1}{c}\left[Q\frac{\partial \phi}{c\, \partial \theta}-\phi\frac{\partial Q}{c\, \partial \theta}\right] = -4\pi\int_{-\infty}^{t'} d\theta\, Q\rho(\xi\eta\zeta\theta)$$

$$\nabla^2\phi-\frac{\partial^2\phi}{c^2\,\partial t^2}-k^2\phi=-4\pi\rho(xyzt).$$

or $$\sum\frac{\partial}{\partial\xi}\int_{-\infty}^{t'}d\theta\left(Q\frac{\partial\phi}{\partial\xi}-\phi\frac{\partial Q}{\partial\xi}\right)+\sum\left[Q\frac{\partial\phi}{\partial\xi}-\phi\frac{\partial Q}{\partial\xi}\right]\frac{1}{c}\frac{\partial r}{\partial\xi}-\frac{1}{c}\left[Q\frac{\partial\phi}{c\,\partial\theta}-\phi\frac{\partial Q}{c\,\partial\theta}\right]$$

$$=-4\pi\int_{-\infty}^{t'}d\theta\;Q\rho(\xi\eta\zeta\theta).$$

Rearranging the terms in this equation we write

$$\sum\frac{\partial}{\partial\xi}\int_{-\infty}^{t'}d\theta\left(Q\frac{\partial\phi}{\partial\xi}-\phi\frac{\partial Q}{\partial\xi}\right)-\frac{[\phi]}{c}\left[\sum\frac{\partial Q}{\partial\xi}\frac{\partial r}{\partial\xi}-\frac{\partial Q}{c\,\partial\theta}\right]+\frac{[Q]}{c}\left[\sum\frac{\partial\phi}{\partial\xi}\frac{\partial r}{\partial\xi}-\frac{\partial\phi}{c\,\partial\theta}\right]$$

$$=-4\pi\int_{-\infty}^{t'}d\theta\,Q\rho(\xi\eta\zeta\theta)\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad(19)$$

and observe that

$$\left[\frac{\partial Q}{\partial\theta}\right]=0=\left[\frac{\partial Q}{\partial\xi}\right]=\left[\frac{\partial Q}{\partial\eta}\right]=\left[\frac{\partial Q}{\partial\zeta}\right],$$

$$[Q]=\frac{k}{2},$$

and, in view of eqs. (17),

$$\left[\sum\frac{\partial\phi}{\partial\xi}\frac{\partial r}{\partial\xi}-\frac{\partial\phi}{c\,\partial\theta}\right]=\sum\frac{\partial r}{\partial\xi}\frac{\partial}{\partial\xi}[\phi]$$

$$=\sum\frac{\partial}{\partial\xi}\left(\frac{\partial r}{\partial\xi}[\phi]\right)-\frac{2}{r}[\phi].$$

Equation (19) now takes the form

$$\sum\frac{\partial}{\partial\xi}\left\{\int_{-\infty}^{t'}d\theta\left(Q\frac{\partial\phi}{\partial\xi}-\phi\frac{\partial Q}{\partial\xi}\right)+\frac{k}{2c}\frac{\partial r}{\partial\xi}[\phi]\right\}-\frac{k}{c}\frac{[\phi]}{r}$$

$$=-4\pi\int_{-\infty}^{t'}d\theta\,Q\rho(\xi\eta\zeta\theta).\quad\ldots\quad\ldots\quad\ldots\quad(B)$$

Combining now equations (A) and (B) we get

$$\sum\frac{\partial}{\partial\xi}\left\{\frac{1}{r}\frac{\partial}{\partial\xi}[\phi]-[\phi]\frac{\partial}{\partial\xi}\left(\frac{1}{r}\right)+\frac{2}{c}\frac{1}{r}\frac{\partial r}{\partial\xi}\frac{\partial}{\partial t}[\phi]-ck\int_{-\infty}^{t'}d\theta\left(Q\frac{\partial\phi}{\partial\xi}-\phi\frac{\partial Q}{\partial\xi}\right)\right.$$

$$\left.-\frac{k^2}{2}\frac{\partial r}{\partial\xi}[\phi]\right\}$$

$$=-4\pi\left\{\frac{[\rho]}{r}-ck\int_{-\infty}^{t'}d\theta\,Q\rho(\xi\eta\zeta\theta)\right\}$$

or, in view of eqs. (17),

$$\sum \frac{\partial}{\partial \xi}\left\{\frac{1}{r}\left[\frac{\partial \phi}{\partial \xi}\right]+\frac{1}{r}\frac{\partial r}{\partial \xi}\left[\frac{\partial \phi}{c\,\partial \theta}\right]-[\phi]\frac{\partial}{\partial \xi}\left(\frac{1}{r}\right)-ck\int_{-\infty}^{t'} d\theta\left(Q\frac{\partial \phi}{\partial \xi}-\phi\frac{\partial Q}{\partial \xi}\right)\right.$$
$$\left.-\frac{k^2}{2}\frac{\partial r}{\partial \xi}[\phi]\right\}$$
$$=-4\pi\left\{\frac{[\rho]}{r}-ck\int_{-\infty}^{t'} d\theta\, Q\rho(\xi\eta\zeta\theta)\right\}. \quad \ldots \quad \ldots \quad (20)$$

This equation (20) is easily seen to be identical with eq. (12) and subsequent integration through a closed volume, as in the last section, will now obviously lead to the complete solution of the proposed equation in the form (13).

It is evident that we may in like manner obtain the complete solution for advanced values of t if we replace θ in eq. (10) at the outset by $t+\frac{r}{c}$ $(=t'')$ and later perform integrations with respect to θ from t'' to $+\infty$.

6. In closing this communication we like to make a few observations on the methods here employed. Firstly, the method of complex integration is obviously also available for the Kirchhoff solution of the equation

$$\nabla^2\phi-\frac{\partial^2\phi}{c^2\,\partial\theta^2}=-4\pi\rho(\xi\eta\zeta\theta).$$

We may have this solution, of course, immediately out of the solutions (13) and (14) with $k \to 0$; we may have it, too, by a like procedure repeated with a solving function $\chi(s)=\frac{1}{s^2}$. Secondly, both methods may be adapted without difficulty for solution of the equation

$$\nabla^2\phi-\frac{\partial^2\phi}{c^2\,\partial\theta^2}+k^2\phi=-4\pi\rho(\xi\eta\zeta\theta)$$

and the only difference should be that real Bessel functions of imaginary arguments would take the place of those we have used above. Lastly, underlying the method of complex integration there is obviously the assumption that ρ and ϕ are analytic functions of θ over a region of the complex plane wide enough to embrace the path of integration—an assumption from which the second method is happily free.

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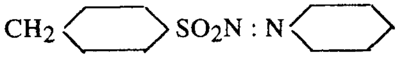
REACTION OF SULPHONAZIDES WITH PYRIDINE : SALTS AND DERIVATIVES OF PYRIDINE-IMINE

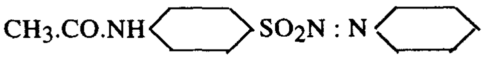
Curtius and Kraemers[1] heated p-toluenesulphonazide with pyridine and isolated a crystalline compound, M.P. 210°C. They supposed it to be p-toluenesulphon—amido-pyridine. When, however, the sulphonic acid residue was removed by hydrolysis, the new base yielded a picrate of M.P. 138-9°C. which did not agree with M.P. of the picrate of any of the known amino-pyridines. The structure of this compound was thus not established by this work of Curtius.

Reactions of p-acetyl-amino-benzenesulphonazide and p-toluene-sulphonazide with pyridine have been studied here for the last year and a half. This has led to the discovery of a new derivative of pyridine, N-imino-pyridine, and the compound of Curtius has now been shown to be a sulphonic acid derivative of this N-amino-pyridine or pyridine-imine.



(I)

When p-acetyl-amino-benzene-sulphonazide is boiled in dry pyridine in an inert atmosphere, nitrogen is slowly evolved and from among other products a crystalline compound insoluble in pyridine can be isolated. After repeated crystallisation the substance melts with decomposition at 283°-4°C. The analysis shows that the substance has the same empirical formula as p-acetyl-amino-benzenesulphonamido-pyridine but closer investigation reveals it to be also a derivative of pyridine-imine having the formula



(II)

The acetyl-group can be removed easily and this process leads to an extremely hygroscopic hydrochloride of a base. The free base has M.P. 228-229°C. Estimation of Pt. in the platini-chloride compound, M.P. 224°C, indicats the formula

$$(C_{11}H_{11}O_2N_3S)H_2PtCl_6$$

showing that it has two basic groups.

When further hydrolysed with hydrochloric acid, sulphanilic acid separates out and the solution of an extremely hygroscopic hydrochloride of a base is obtained.

The perchlorate of this base has M.P. 204°C., the picrate, 149°C. and the platini-chloride, 237°C. and the analyses completely agree with the expected formulae.

Attempts to liberate the free-base with alkali lead to polymerisation.

Alkaline-ferricyanide liberates nitrogen from the salts of this base. Nitrous acid decomposes the salts and from the solution pyridine can be isolated, as perchlorate.

The same base is obtained from the hydrolysis of paratoluene-sulphonazide-compound. This has been confirmed by allowing p-acetyl-amino-benzenesulpho-chloride to react on the base from (I) which leads to the compound (II).

A confirmation of the imino-structure has been obtained by a synthesis of the compound (I), from glutaconic-dialdehyde. Mono-benzoyl-derivative of glutaconic-dialdehyde[2] is allowed to react with paratoluene sulpho-hydrazide leading to the compound of the form

$$C_6H_5COO\ CH = CH - CH = CH - CH = N.NHSO_2C_7H_7$$

With the action of alcoholic hydrochloric acid the benzoyl group is eliminated, and there is a ring closure leading to the formation of the pyridine ring, and the hydrochloride of compound (I). This synthesis is similar to the synthesis of pyridine-oxide by Baumgarten.[3]

Further investigations about this interesting class of compound are in progress. Full details will be published elsewhere.

In April number of *Current Science*, Ganapathi and Miss Alamela[4] reported about their attempts to prepare sulphonamide derivatives of heterocyclic compounds by Curtius process. They mentioned the compound (II) and assumed this to be a 3-amino-pyridine derivative without any justification. It is expected that this note may be of some interest to them.

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Dacca, 1-6-1943.

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A NOTE ON DIRAC EQUATIONS AND THE ZEEMAN EFFECT

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ABSTRACT. A new treatement has been given for solving Dirac's equations for hydrogenic atoms, and the radial functions are expressed in terms of a combination of Sonine's polynomials $T_n^{(\mu)}(\rho)$, $T_{n-1}^{(\mu)}(\rho)$ of only two consecutive degrees n, $n-1$; and the elementary properties of such polynomials have enabled us to tackle the Zeeman effect problem in general (homogeneous field) leading to the standard quadratic equation in energy for the effect.

1. With the help of the two-dimensional matrices s_x, s_y, s_z, of Pauli the wave-equations of Dirac can be put in the well-known matrix form :

$$\frac{\iota}{hc}\left[E+E_0+\frac{Ze^2}{r}\right]X+DY=0,\ \frac{\iota}{hc}\left[E-E_0+\frac{Ze^2}{r}\right]Y+DX=0 \tag{1.1}$$

where D is the operator $\qquad s_x\frac{\partial}{\partial x}+s_y\frac{\partial}{\partial y}+s_z\frac{\partial}{\partial z}.$

If, similarly $S = xs_x+ys_y+zs_z$, and $s = \frac{I}{r}(xs_x+ys_y+zs_y)$, then

$S.D. = r\frac{d}{dr}+L$, where $L = \iota\,(M_xs_x+M_ys_y+M_zs_z)$.

And the following commutation rules can be easily deduced :

$$s(L-1)+(L-I)s = 0,\ D(L-1)+(L-1)D = 0;$$

also $\qquad s^2 \equiv 1.$

Hence multiplying the equations on the left by S we have

$$\frac{\iota}{hc}\left[E+E_0+\frac{Ze^2}{r}\right]SX+\left[r\frac{d}{dr}+1\right]Y+(L-1)Y=0; \tag{1.2}$$

$$\frac{\iota}{hc}\left[E-E_0+\frac{Ze^2}{r}\right]SY+\left[r\frac{d}{dr}+1\right]X+(L-1)X=0.$$

Assuming $X \equiv \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \frac{f(r)}{r}\,\Phi$; $Y \equiv \begin{vmatrix} u_3 \\ u_4 \end{vmatrix} = \frac{g(r)}{r}\,\Psi$ where Φ, ψ are similar one-column matrices and functions of (θ, ϕ), the equations can be rewritten as

$$\frac{\iota}{\hbar c} f(r)\left[E+E_0+\frac{Ze^2}{r}\right]s\Phi+\frac{dg(r)}{dr}\cdot\Psi+\frac{g(r)}{r}(L-1)\Psi=0, \qquad (1.3)$$

$$\frac{\iota}{\hbar c} g(r)\left[E-E_0+\frac{Ze^2}{r}\right]s\Psi+\frac{df(r)}{dr}\cdot\Phi+\frac{f(r)}{r}(L-1)\Phi=0.$$

The equations become easily separable if the matrices Φ and Ψ are so chosen that at first $(L-1)\Psi = k\Psi$ and $s\Phi = \Psi$; and as $s^2 \equiv 1$, it follows therefore from commutation rules that

$$\Phi = s\Psi \text{ and } (L-1)s\Psi = -ks\Psi.$$

We observe at once that $L(L-1)\,\Psi = -\underline{\nabla_2}\Psi = k(k+1)\Psi$, where

$$\nabla_2 \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\;\frac{\partial^2}{\partial\phi^2};$$

and therefore k can be either a positive or a negative integer.

Secondly, if (Φ, Ψ) are the matrices for positive k, then (Ψ, Φ) are the matrices for negative k. Also remembering that the operator

$-\left[\iota\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)+\frac{1}{2}s_z\right]$ commutes with the equation-system, the angular matrices can be expressed in terms of spherical harmonics of order k and $k-1$ in the following form :

$$\Phi \equiv \begin{vmatrix} \sqrt{\dfrac{k+m}{2k-1}}\;Y^{\mu}_{k-1} \\[2ex] \sqrt{\dfrac{k-m-1}{2k-1}}\;Y^{\mu+1}_{k+1} \end{vmatrix},\ \Psi \equiv \begin{vmatrix} \sqrt{\dfrac{k-m}{2k+1}}\;Y^{\mu}_{k} \\[2ex] \sqrt{\dfrac{k+m+1}{2k+1}}\;Y^{\mu+1}_{k} \end{vmatrix} \quad (k\text{, a positive integer}) \qquad (1.4)$$

where $Y^{\mu}_{k} \equiv \sqrt{\frac{2k+1}{2}\cdot\frac{(k-\mu)!}{(k+\mu)!}}\,\frac{1}{2^k k\,!}\,\frac{d^{k+\mu}}{dx^{k+\mu}}\,(x^2-1)^k\,\frac{e^{i\mu\Phi}}{\sqrt{2\pi}}.$ $(x = \cos\theta)$

The functions are interchanged for negative values of k. Writing $f(r) = \iota F(r)$ so as to remove the imaginary from the radial equations, we get

$$\left[\frac{dG}{dr} + k\frac{G}{r}\right] - \frac{1}{hc}\left[E + E_0 + \frac{Ze^2}{r}\right] F = 0,$$

$$\left[\frac{dF}{dr} - k\frac{F}{r}\right] + \frac{1}{hc}\left[E - E_0 + \frac{Ze^2}{r}\right] G = 0. \tag{1.5}$$

2. To solve the equation we assume

$$F = f_0 e^{-\lambda r} r^{\mu} F_1(r),\ G = g_0 e^{-\lambda r} r^{\mu} G_1(r), \text{ and } N = \alpha E/(E_0^2 - E^2)^{\frac{1}{2}} \quad \ldots \tag{2.1}$$

where $Ze^2/hc = \alpha$.

Substituting in (1.5), F_1 and G_1 are easily seen to satisfy the following equations :

$$\frac{dG_1}{dr} - \lambda G_1 + \lambda F_1 + \frac{k+\mu}{r} G_1 + \frac{N}{E}(E_0 - E)\frac{F_1}{r} = 0, \tag{2.2}$$

$$\frac{dF_1}{dr} - \lambda F_1 + \lambda G_1 - \frac{k-\mu}{r} F_1 - \frac{N}{E}(E_0 + E)\frac{G_1}{r} = 0,$$

where $f_0/g_0 = -(E_0 - E)/hc\lambda = \dfrac{\lambda hc}{E_0 + E} = -\left(\dfrac{E_0 - E}{E_0 + E}\right)^{\frac{1}{2}}$ and $\lambda = \left(\dfrac{E_0^2 - E^2}{hc}\right)^{\frac{1}{2}}$ (2.3)

If $F_1 + G_1 = 2\chi^+$, $G_1 - F_1 = 2\chi^-$, elimination leads easily to the following equations :

$$r\frac{d^2\chi^+}{dr^2} + (2\mu + 1 - 2\lambda r)\frac{d\chi^+}{dr} + 2\lambda(N - \mu)\chi^+ = 0, \tag{2.4}$$

$$r\frac{d\chi^+}{dr} - (N - \mu)\chi^+ = \left(N\frac{E_0}{E} - k\right)\chi^-, \tag{2.5}$$

provided $k^2 - \mu^2 = N^2(E_0^2 - E^2)/E^2 = \alpha^2.$

Taking the first equation it can be easily seen that it admits of polynomial solutions if $N - \mu = n$ (an integer). Writing ρ for $2\lambda r$, and writing a Sonine polynomial in the form

$$T_n^{(\mu)}(\rho) = B_n^{(\mu)}\left[\rho^n - \frac{n(n+2\mu)}{1\,!}\rho^{n-1} + \frac{n(n-1)(n+2\mu)(n+2\mu-1)}{2!}\rho^{n-2} - \cdots\right]$$

$$= B_n^{(\mu)} \cdot (-1)^n e^{\rho} \rho^{-2\mu} \frac{d^n}{d\rho^n}\left[e^{-\rho}\rho^{n+2\mu}\right] \tag{2.6}$$

with constant $B_n^{(\mu)}$ so chosen that it is normalized according to

$$\int_0^\infty e^{-\rho}\rho^{2\mu}\, T_n^{(\mu)}(\rho)T_n^{(\mu)}(\rho)d\rho = 1$$

we get $B_n^{(\mu)} = 1/\{n\,!\;\Gamma(n+2\mu+1)\}^{\frac{1}{2}}$

Hence it easily follows that χ^+ and X^- will have solutions as given below :

$$\chi^+ = \delta\left(N\frac{E_0}{E} - k\right)^{\frac{1}{2}} T_n^{(\mu)}(\rho) \tag{2.7}$$

$$\left[\; N = n+\mu = \frac{\alpha\epsilon}{\sqrt{1-\epsilon^2}}\;;\; \epsilon = \frac{E}{E_0}\;\right]$$

$$\chi^- = \delta\left(N\frac{E_0}{E} + k\right)^{\frac{1}{2}} T_{n-1}^{(\mu)}(\rho) \tag{2.8}$$

where the normalising factor δ is to be determined presently.

Our radial functions of eqs. (2.1) stand thus

$$F = -\delta\sqrt{\left(1-\frac{E}{E_0}\right)}\; e^{-\frac{\rho}{2}}\rho^\mu\left[\left(N\frac{E_0}{E}-k\right)^{\frac{1}{2}}T_n^{(\mu)}(\rho) - \left(N\frac{E_0}{E}+k\right)^{\frac{1}{2}}T_{n-1}^{(\mu)}(\rho)\right] \tag{2.9}$$

$$G = \delta\sqrt{\left(1+\frac{E_0}{E}\right)}\; e^{-\frac{\rho}{2}}\rho^\mu\left[\left(N\frac{E_0}{E}-k\right)^{\frac{1}{2}}T_n^{(\mu)}(\rho) + \left(N\frac{E_0}{E}+k\right)^{\frac{1}{2}}T_{n-1}^{(\mu)}(\rho)\right] \tag{2.10}$$

Normalisation requires

$$\int_0^\infty (F^2+G^2)dr = 1,$$

which gives $$\delta = \sqrt{\frac{\lambda}{2N}\frac{E}{E_0}}\,.$$

We can write the two solutions (A) and (B), corresponding to positive and negative values of k respectively.

$$\left[\begin{array}{ll} u_1 = \sqrt{\dfrac{k+\mu}{2k-1}}\;.\iota\dfrac{f_+(r)}{r}\,Y_{k-1}^{\mu}, & u_3 = \sqrt{\dfrac{k-\mu}{2k+1}}\,\dfrac{g_+(r)}{r}\,Y_k^{\mu} \\[2ex] u_2 = -\sqrt{\dfrac{k-\mu-1}{2k-1}}\;\iota\dfrac{f_+(r)}{r}\,Y_{k-1}^{\mu+1}, & u_4 = \sqrt{\dfrac{k+\mu+1}{2k+1}}\,\dfrac{g_+(r)}{r}\,Y_k^{\mu+1} \end{array}\right] \tag{A}$$

$$\left[\begin{array}{ll} v_1 = \sqrt{\frac{k-\mu}{2k+1}}\,\iota\frac{f_-(r)}{r}\,Y_k^\mu & v_3 = \sqrt{\frac{k+\mu}{2k-1}}\,\frac{g_-(r)}{r}\,Y_{k-1}^\mu \\ v_2 = \sqrt{\frac{k+\mu+1}{2k+1}}\,\iota\frac{f_-(r)}{r}\,Y_k^{\mu+1}, & v_4 = -\sqrt{\frac{k-\mu-1}{2k-1}}\,\frac{g_-(r)}{r}\,Y_{k-1}^{\mu+1} \end{array}\right] \tag{B}$$

wherein it is understood that f_- and g_- are obtained by changing the sign of k, in the expressions for F and G in (1.5) above.

3. When the atom is in an electromagnetic field defined by the vector potential $(A_xA_yA_z)$, the Dirac-equations become

$$\frac{\iota}{hc}\left[E+E_0+\frac{Ze^2}{r}\right]X+DY+\frac{e}{hc}AY=0 \tag{3.1}$$

$$\frac{\iota}{hc}\left[E-E_0+\frac{Ze^2}{r}\right]Y+DX+\frac{e}{hc}AX=0 \tag{3.2}$$

where $$A \equiv \iota[A_xs_x+A_ys_y+A_zs_z]$$

In the case of a constant magnetic field H, in the direction of Z-axis

$$A \equiv \tfrac{1}{2}Hr\begin{vmatrix} 0 & e^{-i\mu\phi}\sin\theta \\ -e^{i\mu\phi}\sin\theta & 0 \end{vmatrix} \equiv \tfrac{1}{2}Hra. \tag{3.3}$$

We observe in passing that in the absence of A, the wave function X is generally small compared with Y—the radial component of $F(r)$ has the factor $\sqrt{1-\frac{E}{E_0}}$, while $G(r)$ has $\sqrt{1+\frac{E}{E_0}}$; so that the perturbation effect of $\frac{e}{hc}AX$ is small as compared with effect due to the term $\frac{e}{hc}AY$ in eq. (3.1)

Remembering

$$e^{-i\phi}Y_k^{\mu+1}\sin\theta = -\sqrt{\frac{(k-\mu)(k-\mu+1)}{(2k+1)(2k+3)}}\,Y_{k+1}^\mu + \sqrt{\frac{(k+\mu)(k+\mu+1)}{(2k+1)(2k-1)}}\,Y_{k-1}^\mu\,;$$

$$e^{i\phi}Y_k^\mu\sin\theta = \sqrt{\frac{(k+\mu+1)(k+\mu+2)}{(2k+1)(2k+3)}}\,Y_{k+1}^{\mu+1} - \sqrt{\frac{(k-\mu)(k-\mu-1)}{(2k-1)(2k+1)}}\,Y_{k-1}^{\mu+1}\,;$$

we seek an approximate solution of the equation by choosing one set of angular function $(Y^{\mu}_{k},\ Y^{\mu+1}_{k})$, and assuming the existence of both the sets $(Y^{\mu}_{k-1},\ Y^{\mu+1}_{k-1})$ and $(Y^{\mu}_{k+1},\ Y^{\mu+1}_{k+1})$ in X. This can be done by suitably combining the (A) and (B) types, thus

Take $u_j = C^{+}_{l} u^{+}_{j} + C^{-}_{-(l+1)} v^{-}_{j}$, $(j = 1, 2, 3, 4)$, where $(u^{+}_{1}, u^{+}_{2}, u^{+}_{3}, u^{+}_{4})$ corresponds to $k = +l$ and $(v^{-}_{1}, v^{-}_{2}, v^{-}_{3}, v^{-}_{4})$ corresponds to $k = -(l+1)$. More explicitly, their values are

$$u^{+}_{1} = \sqrt{\frac{l+\mu}{2l-1}}\ \iota\frac{f_{+}(r)}{r} Y^{\mu}_{l-1}; \qquad v^{-}_{1} = \sqrt{\frac{l-\mu+1}{2l+3}}\ Y^{\mu}_{l+}\ \iota\frac{f_{-}(r)}{r};$$

$$u^{+}_{1} = -\sqrt{\frac{l-\mu-1}{2l-1}}\ \iota\frac{f_{+}(r)}{r}\ Y^{\mu+1}_{l-1}; \quad v^{-} = \sqrt{\frac{l+\mu+2}{2l+3}}\ Y^{\mu+1}_{l+1}\,\iota\frac{f_{-}(r)}{r};$$

$$u^{+}_{3} = \sqrt{\frac{l-\mu}{2l+1}}\ \frac{g_{+}(r)}{r}\ Y^{\mu}_{l}; \qquad v^{-}_{3} = \sqrt{\frac{l+\mu+1}{2l+1}}\ \frac{g_{-}(r)}{r}\ Y^{\mu}_{l};$$

$$u^{+}_{1} = \sqrt{\frac{l+\mu+1}{2l+1}}\ \frac{g_{+}(r)}{r} Y^{\mu+1}_{l}; \quad v^{-}_{4} = -\sqrt{\frac{l-\mu}{2l+1}}\ \frac{g_{-}(r)}{r}\ Y^{\mu+1}_{l}.$$

The constants λ, N, μ of the two types of solutions are different and are expressed by the following relations

$$\lambda_{+} = (E^{2}_{0}-E^{2}_{+})^{\frac{1}{2}}/hc, \quad l^{2}-\alpha^{2} = \mu^{2}_{+}, \quad N_{+} = \alpha E_{+}/[E^{2}_{0}-E^{2}_{+}]^{\frac{1}{2}}, \tag{3.4}$$

and $\quad N_{+} = n_{+}+\mu_{+}$[for (A)—type with $k = l$].

$$\lambda_{-} = (E^{2}_{0}-E^{2}_{-})^{\frac{1}{2}}/hc), \quad (l+1)^{2}-a^{2} = \mu^{2}_{-}, \quad N_{-} = aE_{-}/[E^{2}_{0}-E^{2}_{0}]^{\frac{1}{2}}, \tag{3.5}$$

and $\quad N_{-} = n_{-}+\mu_{-}$[for (B)—type with $k = -(l+1)$].

If E_{+} and E_{-} differ slightly from one another, we have the following approximate relations

$$\lambda_{+} = \lambda_{-}, \ \mu_{-} = \mu_{+}+1, \ N_{+} = N_{-}, n_{-} = n_{+}-1, \text{ and } \frac{E_{+}}{E_{0}} \sim 1. \tag{3.6}$$

By following the usual method of perturbation we see easily that the characteristic equation for determining the Eigen value E would be

$$\begin{vmatrix} \frac{\iota}{h_c}\left[E-E_{+}\right]+pa_{11} & pa_{12} \\ pa_{21} & \frac{\iota}{h_c}\left[E-E_{-}\right]+pa_{22} \end{vmatrix} = 0 \tag{3.7}$$

where $p = eH/2hc$, and

$$
\begin{aligned}
\boldsymbol{a}_{11} &= \int [\bar{X}^{*}_{+} r\boldsymbol{a} Y_{+} + Y^{*}_{+} r\boldsymbol{a} X_{+}] dV \\
\boldsymbol{a}_{22} &= \int [\bar{X}^{*}_{-} r\boldsymbol{a} Y_{-} + \bar{Y}^{*}_{-} r\boldsymbol{a} X_{-}] dV \\
\boldsymbol{a}_{12} &= \int [\bar{X}^{*}_{+} r\boldsymbol{a} Y_{-} + \bar{Y}^{*}_{-} r\boldsymbol{a} X_{-}] dV \qquad (3.8) \\
\boldsymbol{a}_{21} &= \int [\bar{X}^{*}_{-} r\boldsymbol{a} Y_{+} + \bar{Y}^{*}_{-} r\boldsymbol{a} X_{+}] dV
\end{aligned}
$$

and $\bar{X}^{*}_{+}$ etc. are transposed conjugates of X_{+} (the $\boldsymbol{a}$ is given in (3.3))

Carrying out integrations over the angle-variable-space we get

$$
\begin{aligned}
a_{11} &= -\frac{2\iota(4\mu+2)l}{(2l-1)(2l+1)} \int_{0}^{\infty} r f_{+} g_{+} dr \; ; \\
a_{22} &= \frac{2\iota(4\mu+2)(l+1)}{(2l+1)(2l+3)} \int_{0}^{\infty} r f_{-} g_{-} dr \; ; \\
a_{12} = a_{21} &= \frac{\iota\sqrt{l(-\mu)(l+\mu+1)}}{2l+1} \int_{0}^{\infty} r(f_{-}g_{+} + f_{+}g_{-})\, dr.
\end{aligned}
$$

4. To evaluate the integral {I} or {II}, we substitute the values of the corresponding f, g, and remembering $r = \rho/2\lambda$, we obtain

$$
\begin{aligned}
\{\mathrm{I}\} = -\frac{\lambda_{+}}{2N_{+}} \frac{E_{+}}{E_{0}^{2}} \frac{(E_{0}^{2}-E_{+}^{2})^{\frac{1}{2}}}{4\lambda^{2}} \Bigg\{ \left(N_{+} \frac{E_{0}}{E_{+}} - k \right) \int_{0}^{\infty} e^{-\rho} \rho^{2\mu+1} \left[T_{n}^{(\mu)}(\rho) \right]^{2} d\rho \\
- \left(N_{+} \frac{E_{0}}{E_{+}} + k \right) \int_{0}^{\infty} e^{-\rho} \rho^{2\mu+1} \left[T_{n-1}^{(\mu)}(\rho) \right]^{2} d\rho \Bigg\}.
\end{aligned}
$$

The integrals are evaluated quickly by repeated partial integration; thus

$$
\int_{0}^{\infty} e^{-\rho} \rho^{2\mu+1} \Big] \; T_{n}^{(\mu)}(\rho) \; \Big]^{2} d\rho = \frac{(-)^{n}}{\sqrt{n\,!\,\Gamma(n+2\mu+1)}} \int_{0}^{\infty} \rho T_{n}^{(\mu)}(\rho) \frac{d^{n}}{d\rho} \left[e^{-\rho} \rho^{2\mu+n} \right] d\rho
$$

$$
= \int_{0}^{\infty} \frac{e^{-\rho}}{n!} \frac{\rho^{2\mu+n}}{\Gamma(n+2\mu+1)} d\rho . \frac{d^{n}}{d\rho^{n}} \left[\rho \left(\rho^{n} - \frac{n(n+2\mu)}{1!} \rho^{n-1} + \dots \right) \right]
$$

$$
= (n+1)(n+2\mu+1) - n(n+2\mu) = 2N_{+} + 1.
$$

Similarly

$$\int_0^\infty e^{-\rho}\rho^{2\mu+1}\Big[\ T^{(\mu)}_{n-1}\ (\rho)\Big]^2 d\rho = 2N_+-1.$$

An easy substitution of the value of λ_+ in the foregoing leads to the following results :

$$\int_0^\infty rf_+g_+dr = -\frac{h_c}{4E_0}\left(1-2l\,\frac{E_0}{E_+}\ \right); \tag{4.1}$$

$$\int_0^\infty rf_-g_-dr = -\frac{h_c}{4E_0}\left(1+2(l+1)\,\frac{E_0}{E_-}\ \right). \tag{4.2}$$

The evaluation of the third integral cannot, however, be exactly expressed in a neat form; for the simple reason that λ_+, N_+, E_+ as well as λ_-, N_-, E_- are different in the two solutions, as also $\rho_+ = 2\lambda_+r$ and $\rho_- = 2\lambda_-r$. If however, we introduce approximations in the beginning, and take

$$\lambda_+ = \lambda_- = |\lambda|,\ E_+ = E_-,\ \ \mu_- = \mu_++1,\ \ \ n_- = n_+-1,\ \ \ \rho_+ = \rho_-, \tag{4.3}$$

we see the third integral

$$\{\mathrm{III}\} = -\frac{2|\lambda|}{2N}\ \frac{|E|}{E_0^2}\,\frac{(E_0-|E|^2)^{\frac{1}{2}}}{4|\lambda|^2}\Big\{\Big(\ N\frac{E_0}{E}-l\Big)^{\frac{1}{2}}\Big(\ N\frac{E_0}{E}+l+1\ \Big)^{\frac{1}{2}}\times$$
$$\int_0^\infty e^{-\rho}\rho^{2\mu+2}T^{(\mu)}_{n}(\rho)T^{(\mu+1)}_{n-1}\,(\rho)d\rho-\Big(\ N\frac{E_0}{E}+l\Big)^{\frac{1}{2}}\Big(\ N\frac{E_0}{E}-l-1\ \Big)^{\frac{1}{2}}$$
$$\times\int_0^\infty e^{-\rho}\rho^{2\mu+2}T^{(\mu)}_{n}(\rho)\,T^{(\mu+2)}_{n-2}\,(\rho)d\rho\Big\}.$$

The two definite integrals can be easily evaluated in the same way as before. We give the results below ;

$$\int_0^\infty e^{-\rho}\rho^{2\mu+2}\ T^{(\mu)}_{n}(\rho)\,T^{(\mu+1)}_{n-1}\ (\rho)d\rho = \int_0^\infty \frac{e^{-\rho}\rho^{n+2\mu}}{\sqrt{n-1!n\ !\ \Gamma(n+2\mu+1)\ \Gamma(n+2\mu+2)}} \tag{4.4}$$
$$\times\ \frac{d^n}{d\rho^n}\Big[\ \rho^{n+1}-(n-1)(n+2\mu+1)\rho^n+\dots\ \Big] = 2\sqrt{(N-l)(N+l+1)}.$$

$$\int_0^\infty\ e^{-\rho}\rho^{2\mu+2}T^{(\mu)}_{n-2}(\rho)T^{(\mu+1)}_{n-}\,d\rho =\ 2\sqrt{(N+l)(N-l-1)}. \tag{4.5}$$

Hence

$$\{III\} = -\frac{1}{2N}\frac{\boldsymbol{h}_c}{E_0}\Big[\sqrt{N\frac{E_0}{E}-l}\,\sqrt{N\frac{E_0}{E}+l+1}\,\sqrt{N-l}\,\sqrt{N+l+1}$$

$$-\sqrt{N\frac{E_0}{E}+l}\,\sqrt{N\frac{E_0}{E}-l-1}\,\sqrt{N+l}\,\sqrt{N-l-1}\;\Big].$$

Observing $E_0/E \sim 1$, we see

$$\{III\} = -\frac{1}{2N}\cdot\frac{\boldsymbol{h}_c}{E_0}2N = -\frac{\boldsymbol{h}_c}{E_0}.$$

Finally, putting $p = eH/2\boldsymbol{h}_c$, and $e\boldsymbol{h}/2m_0c = \mu_0$ (Bohr-magneton) and substituting the values of the three integrals (and making $E_0/E \sim 1$) we see that the determinantal equation (3.7) takes the form :

$$\begin{vmatrix} E-E_{+}-\frac{2l}{2l+1}\,m\mu_0 H & -\frac{\mu_0 H}{2l+1}\sqrt{(l-m+\frac{1}{2})(l+m+\frac{1}{2})} \\ -\frac{\mu_0 H}{2l+1}\sqrt{(l-m+\frac{1}{2})(l+m+\frac{1}{2})} & E-E_{-}-\frac{2l+2}{2l+1}\,m\mu_0 H \end{vmatrix} = 0 \qquad (4.6)$$

wherein we have put $\mu+\frac{1}{2} = m$ (magnetic quantum number).

The result (4.6) agrees completely with that quoted by Bethe (1933), supposed to have been worked out by him from Pauli's equations. Condon and Shortley (1935) obtained similar determinant from principles of quantum mechanics by applying two-fold perturbations (spin-orbit and magnetic) simultaneously. It may be noted that Darwin (1928) has many similar features with our mode of attack, and we can claim some elegance by our introduction of Sonines properties, which exhibit our solutions in a good perspective.

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Thirtyfirst Indian Science Congress

DELHI, 1944

PRESIDENTIAL ADDRESS

CONGRESS PRESIDENT : PROFESSOR S. N. BOSE

THE CLASSICAL DETERMINISM AND THE QUANTUM THEORY

(Delivered on Jan. 3, 1944)

I wish to express sincere thanks for the great honour you have done me. The Presidentship of the Science Congress is a great distinction, and I confess, I have my own misgivings about the wisdom of your choice. Your first decision had raised high hopes. Many of us expected that a deliberate programme of the future scientific activities of the country would probably be a feature of the opening speech of this Congress. Pandit Jawaharlal had studied the needs of the country. Many of our front-rank scientists and industrialists had met under his leadership, not long ago, and given to questions of future reconstruction much time and anxious thought. The results of this deliberation would have been invaluable at the present moment. My regret is keen that chance has deprived us of the benefits of a sustained and careful study of the problems of the day. I would have liked to present here the results, if they were available. Unfortunately they are not, as most of the reports are inaccessible to me.

One of your former Presidents had remarked that "a scientist is apt to become a man that knows more and more about less and less, so that his opinion upon subjects outside his field of special study is not necessarily of special value". I realise the wisdom of this warning and hope to have your indulgence, if I seem to be more at home with doubts and criticisms than with useful knowledge.

I would like to present before you certain aspects of modern physics and draw your attention to the profound changes in the principle of scientific explanation of natural phenomena brought about by the quantum theory. The last fifty years record remarkable discoveries. I need only mention the electron and the neutron, X-rays and Radio-activity to remind you of the increase of our knowledge. Our equipment has gained in power range and accuracy. We possess powerful telescopes to scan the furthest corners of the universe, also precise and delicate instruments to probe into the interior of the atoms and molecules. The alchemists' dream of transmutation has become a reality. Atoms are now disintegrated and synthesised. X-ray reveals invisible worlds and wireless links up the furthest ends of the earth with possibility of immediate intercommunication. These discoveries have their repercussions in the realm of ideas. Fifty years ago the belief in causality and determination was absolute. To-day physicists have gained knowledge but lost their faith. To understand properly the significance of such a profound change it will be necessary to discuss briefly how it all came about. Classical physics had begun with the study of astronomy. With his laws of gravitation and his dynamics Newton had explained planetary motion. Subsequent study has shown astronomical prediction to be possible and sure. Physicists had taken the equations of celestial mechanics as their model of a universal law. The atomic theory had in the mean time gained universal acceptance ; since matter had resolved into a conglomeration of particles, the ideal scheme was to explain all phenomena in terms of their motions and interactions. It was only necessary to set up a proper set of equations, and to take account of all possible mutual interactions. If the mass, position, and velocity of all the particles were known at any instant, these equations would theoretically enable the physicist to predict the position and motion of every particle at any other subsequent moment.

The phenomena of light did not at first fit into this simple scheme. To regard it as a stream of particles was impossible due to the discovery of interference. Accordingly the wave theory of light was originated by Huyghens and perfected by Maxwell. With the discovery of the electron as a universal constituent of matter, the electromagnetic theory of Maxwell was converted into an electronic theory by Lorentz. To the dynamical laws were added the electromagnetic equations and the two together apparently gave an exact and ideal formulation of the laws of causality. In the forces of interaction henceforth, were to be included not only the gravitational forces but also those interactions which depended on the charge and the motion of the particles. These interactions were brought about by influences which spread out as waves with the velocity of light. They superimposed, interfered and constituted the field of force in the neighbourhood of the particles, modified their motion and were in turn modified by them. The motions of all particles throughout the universe were thus interlocked. These out-going influences also constituted light, invisible radiation, X-rays and wireless waves. Thus a set of universal laws was supposed to have been discovered and we had only to apply them suitably to find explanations of all conceivable natural phenomena. In physical science we do not however always proceed in the above way and turn to the "microscopic" equations whenever we have to explain events. We often study materials *en masse,* consisting of an enormous number of corpuscles, and we use either the principle of the conservation of energy or the laws of thermodynamics to explain their behaviour. These laws were however regarded either as simple consequences of the fundamental equations or as statistical laws derivable from them by a suitable averaging. Though in the latter cases we talk about probabilities and fluctuations, it was more or less a matter of faith to maintain that if it were possible for us to obtain all the necessary data by delicate observations, universal laws would enable us to follow each individual molecule in this intricate labyrinth and we should find in each case an exact fulfilment of the laws and agreement with observation. The above in brief forms an expression of faith of a classical physicist. We see that it involves as necessary consequences, belief in continuity, in the possibility of space-time description of all changes and in the existence of universal laws independent of observers which inexorably determine the course of future events and the fate of the material world for all times.

A few remarks about the general equations will perhaps enable us to follow better the criticisms that have been levelled against the system. The structure of the mechanical equations of particles is different from the field-equations of Maxwell and Lorentz. The principles of conservation of energy and momentum were first discovered as consequences of the mechanical equations. Mass and velocity of the corpuscle furnish means to measure its momentum, and its energy, if we leave aside the potential energy which resides in the field. To maintain the integrity of the principle of conservation, the field must also be considered capable of possessing energy and momentum, which however, being associated with wave-motion, must spread out in all directions with the waves. The transfer of energy from the field to the particles must thus be a continuous process, whereby, a finite change should come about only in a finite interval and the process should theoretically be capable of an exact description in space and time.

Physics being essentially concerned with relations between quantities, these should all be capable of exact measurement. We measure always intervals of time or inter-distance between points, hence the specification of the reference frame is just as important as the units of measure. Newton had not analysed closely the conception of mass and time. This vagueness persisted in the dynamical equations for the particles. The field-equations which form the basis of the wave-theory of light have a different origin. With the discovery of the principle of the least action, a common derivation of both has been attempted. But a difference in the choice of reference frame in the two apparently subsisted. The wave-equations assumed a fixed ether whereas the material laws contemplated a Galilean inertial-frame. An immediate deduction from this distinction was the possibility of measuring the relative velocity of the observer with reference to ether. The experiment of Michelson and Morley showed it to be unrealisable in practice and formed the starting point of the celebrated Relativity Theory. Einstein had subjected the conception of time-measurement to a searching examination and showed the impossibility of conceiving a time independent of an observer, or an absolute simultaneity of events happening at two different places. The same space-time reference should be chosen for the dynamical equations as well as the equations of the field, this being supplied by the observer. In spite of this apparent limitation Einstein demonstrated the possibility of formulation of natural laws independent of all axes of reference and pointed out that the necessary auxiliaries existed already in the invariant theory and the tensor Calculus of mathematicians. In spite of its apparently revolutionary character, the theory of relativity upheld the ideal of causality and determinism. Einstein himself has continued to seek with great earnestness a unifying field theory which will combine gravitation and electromagnetism and render unnecessary a separate formulation of the dynamical equations. No such theory as yet exists.

II

The development of the quantum theory has raised fundamental issues. Facts have been discovered which demonstrate the breakdown of the fundamental equations which justified our belief in determinism. A critical examination of the way in which physical measurements are made has shown the impossibility of measuring accurately all the quantities necessary for a space-time description of the motion of the corpuscles.

Experiments reveal either the corpuscular or the wave nature for the photon or the electron according to the circumstances of the case, and present us with an apparently impossible task of fusing two contradictory characters into one sensible image. The only solution suggested has been a renunciation of space-time representation of atomic phenomena and with it our belief in causality and determinism.

Let me briefly recapitulate the facts. In 1900 Planck discovered the quantum of action while studying the conditions of equilibrium between matter and the radiation field. Apparently interchange of energy took place in discrete units whose magnitude depended on 'h' and the frequency of the radiation emitted or absorbed by matter. Photo-electric emission had similar disquieting features. Einstein therefore suggested a discrete structure of the radiation field in which energy existed in quanta instead of being continuously distributed in space as required by the wave-theory. This light-quantum however is not the old light-corpuscle of Newton. The rich experimental materials supporting the wave-theory preclude that possibility altogether. Moreover the fundamental relation, $E=h\nu$, and $p=hk$, connecting energy and momentum of the photon with the frequency ν and the vector wave number k, makes a direct reference to idealised plane wave so foreign to the old idea of a corpuscle. Soon afterwards Bohr postulated the existence of radiationless stationary states of atoms and showed how it led to a simple explanation of the atomic spectra. The extreme simplicity of the proposed structure and its striking success in correlating a multitude of experimental facts at once revealed the inadequacy of the ordinary laws of mechanics and electro-dynamics in explaining the remarkable stability of the atoms.

The new ideas found application in different branches of physics. Discontinuous quantum processes furnished solutions to many puzzles. Suitably modified, the theory furnished a reasonable explanation of the periodic classification of elements and thermal behaviour of substances at low temperature. There was however one striking feature. It was apparently impossible to characterise the details of the actual transition processes from one stationary state to another, that is, to visualise it as a continuous sequence of changes determined by any law as yet undiscovered. It became clear that the dynamical laws as well as the laws of electromagnetism failed to account for atomic processes. New laws had to be sought out compatible with the quantum theory capable at the same time of explaining the rich experimental materials of classical physics. Bohr and his pupils utilised for a time a correspondence principle, guessing correct laws for atomic processes from analogy with the results of the classical theory. In every case these appeared as statistical laws concerned with the probabilities of transition between the various atomic states. Einstein tackled the problem of the equilibrium of matter and radiation on the basis of certain hypotheses regarding the probabilities of transition between the various states by absorption and emission. A derivation of the Planck Law was obtained by Bose by a suitable modification of the methods of classical statistics. Heisenberg finally arrived at a satisfactory solution and discovered his matrix-mechanics and a general method for all atomic problems. Dirac and Schrödinger also published simultaneously their independent solutions. Though clothed in apparently dissimilar mathematical symbols, the three theories gave identical results and have now come to be looked upon as different formalisms expressing the same statistical laws.

I have mentioned that the photon gave a simple explanation of many of the properties of radiation and thereby presented its corpuscular aspect while the well-known properties of interference and superposibility brought out its wave character. That the same dual nature may exist in all material corpuscles was first imagined by De Broglie. His phase-waves found quick experimental verification, and raised a similar problem of the real nature of the corpuscle. The formulation of wave-mechanics by Schrödinger, once raised a hope that by a radical modification of our usual ideas about the corpuscle it might be possible to re-establish the law of causality and classical determinism. Subsequent developments have shown such hopes to be illusory. His waves are mathematical fictions utilising the multidimensional representation of a phase-space and are just as incapable of explaining the individuality of the electron, as the photon is incapable of explaining the superposibility of the field. The true meaning of his equations appears in their statistical interpretation.

III

The adherents of the quantum theory interpret the equations in a peculiar way. They maintain that these equations make statements about the behaviour of a simple atom and nothing more than a calculation of the probabilities of transition between its different states is ever possible. There is nothing incomprehensible about such a statistical law even if it relates to the behaviour of a single particle. But a follower of determinism will interpret such statements as betraying imperfect knowledge, either of the attendant circumstances or of the elementary laws. We may record the throws when a certain die is cast a large number of times and arrive at a statistical law which will tell us how many times out of a thousand it will fall on a certain side. But if we can take into account the exact location of its centre of gravity, all the circumstances of the throw, the initial velocity, the resistance of the table and the air and every other peculiarity that may affect it, there can be no question of chance, because each time we can reckon where the die will stop and know in what position it will rest. It is the assertion of the impossibility of even conceiving such elementary determining laws for the atomic system that is disconcerting to the classical physicist.

Von Neumann has analysed the statistical interpretation of the quantum mechanical laws and claims to have demonstrated that the results of the quantum theory cannot be regarded as obtainable from exact causal laws by a process of averaging. He asserts definitely that a causal explanation of quantum mechanics is not possible without an essential modification or sacrifice of some parts of the existing theory.

Bohr has recently analysed the situation and asserted that we cannot hope any future development of the theory will ever allow a return to a description of the atomic phenomena more conformable to the ideal of causality. He points out the importance of the searching analysis of the theory of observation made by Heisenberg, whereby he has arrived at his famous principle of indeterminacy. According to it, it is never possible for us to determine the simultaneous values of momentum, and positional co-ordinates of any system with an accuracy greater than what is compatible with the inequality $\Delta \bar{p}\, \bar{\Delta} q > \frac{h}{4\pi}$.

This natural limitation does not affect the physics of bodies of finite size but makes space-time descriptions of corpuscles and photons impossible. When we proceed to study the behaviour of the elementary particles, our instruments of measurement have an essential influence on the final results. We have also to concede that the contributions of the instrument and the object, are not separately computable from the results as they are interpreted in a classical way with the usual ideas of co-ordinate and momentum accepting thereby a lack of control of all action and reaction of object and instrument due to quantum effects.

It is in this imperative necessity of describing all our knowledge with the usual classical ideas, that Bohr seeks an explanation of the apparently irreconcilable behaviour of corpuscles and radiation in different experiments. For example, if we set our experiments in such a fashion as to determine accurately the space-time co-ordinates, the same arrangement cannot be simultaneously used to calculate the energy momentum relations accurately ; when our arrangements have pushed the accuracy of determining the positional co-ordinates to its utmost limit, the results evidently will be capable only of a corpuscular representation. If, on the other hand, our aim is to determine momentum and energy with the utmost accuracy, the necessary apparatus will not allow us any determination of positional co-ordinates and the results we obtain can be understood only in terms of the imagery of wave-motion. The apparently contradictory nature of our conclusions is to be explained by the fact, that every measurement has an individual character of its own. The quantum theory does not allow us to separate rigorously the contribution of the object and the instrument and as such the sum total of our knowledge gained in individual cases cannot be synthesised to give a consistent picture of the object of our study which enables us to predict with certainty its behaviour in any particular situation. We are thus doomed to have only statistical laws for these elementary particles and any further development is not likely to affect these general conclusions.

It is clear that a complete acceptance of all the above conclusions would mean a complete break with the ancient accepted principles of scientific explanation. Causality and the universal laws are to be thrown simultaneously overboard. These assertions are so revolutionary that, no wonder, they have forced physicists to opposing camps. There are some who look upon causality as an indispensable postulate for all scientific activities. The inability to apply it consistently because of the limitations of the present state of human knowledge would not justify a total denial of its existence. Granted that physics has outgrown the stage of a mechanistic formulation of the principle, they assert that it is now the task of the scientists to seek for a better formulation. Others of the opposing camp look upon old determinism as an inhuman conception, not only because it sets up an impossible ideal, but also as it forces man to a fatalistic attitude which regards humanity as inanimate automata in the hands of an iron law of causation. For them the new theory has humanised physics. The quantum statistical conception of determinism nestles closer to reality and substitutes a graspable truth for an inaccessible ideal. The theory has brought hope and inspired activity. It constitutes a tremendous step towards the understanding of nature. The features of the present theory may not all be familiar but use will remove the initial prejudice. We are not to impose our reason and philosophy on nature. Our philosophy and our logic evolve and adjust themselves more and more to reality.

In spite of the striking successes of the new theory, its provisional character is often frankly admitted. The field theory is as yet in an unsatisfactory state. In spite of strong optimism, difficulties do not gradually dissolve and disappear. They are relegated to a lumber room, whence the menace of an ultimate divergence of all solutions neutralises much of the convincing force of imposing mathematical symbols. Nor is the problem of matter and radiation solved by the theory of complementary characters. Also we hear already of the limitations of the new theory encountered in its application to nuclear problems.

The quantum theory is frankly utilitarian in its outlook ; but is the ideal of a universal theory completely overthrown by the penetrating criticism of the nature of physical measurements?

Bohr has stressed the unique character of all physical measurements. We try to synthesise their results and we get probabilities to reckon with instead of certainties. But how does the formalism $\frac{h}{2\pi i}\frac{\partial\psi}{\partial t}=H\psi$ emerge as a certain law? The wider the generalisation, the less becomes the content. A universal law would be totally devoid of it. It may nevertheless unfold unsuspected harmonies in the realm of concept. More than ever now, physics does need such a generalisation to bring order in its domain of ideas.

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ON AN INTEGRAL EQUATION ASSOCIATED WITH THE EQUATION FOR HYDROGEN ATOM

By
S. N. Bose

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I

The Schrödinger functions 'ϕ' characterising the stationary states of hydrogen atom are now very familiar things in analysis as also the differential equation which they satisfy, namely,

$$\nabla^2\phi+\frac{8\pi^2 m}{h^2}\left(E+\frac{e^2}{r}\right)\phi=0. \qquad \ldots \quad (1,1)$$

The associated functions M, defined by the relation,

$$\phi=\int M \exp.\, 2\pi\iota(lx+my+nz)dl\,dm\,dn, \qquad \ldots \quad (1,2)$$

can be utilised for defining the probability in momentum space if the momentum variables are introduced by the relation $p_x=hl$, $p_y=hm$, $p_z=hn$.

When inversion is possible, (1,2) implies also

$$M(l,m,n)=\int \phi \exp.-2\pi\iota(lx+my+nz)dx\,dy\,dz \qquad \ldots \quad (1,3)$$

and solutions of (1,1) can be utilised to calculate M's. Elsasser (1933) has followed this method and arrived at fairly complicated formula. Another alternative would be to set up an appropriate equation for them, and investigate its solutions. This is an integral equation, whose complete solution is presented here. The analysis presents several interesting features, and leads to expressions of M's, which can be immediately utilised to study their properties or to apply them to physical problems.

II

If we use the semi-convergent integral

$$\frac{1}{R}=\frac{1}{\pi}\int\frac{dl\,dm\,dn}{l^2+m^2+n^2}\exp.\,2\pi\iota(lx+my+nz), \quad R^2=x^2+y^2+z^2, \qquad \ldots \quad (2,1)$$

then combination with (1,2) leads to the following sex-tuple integral for ϕ/R, after a change of variable and order of integration :

$$\frac{\phi}{R}=\frac{1}{\pi}\int\frac{M(\lambda-l,\mu-m,\nu-n)}{l^2+m^2+n^2}\exp.\,2\pi\iota(\lambda x+\mu y+\nu z)dl\,dm\,dn\,d\lambda\,d\mu\,d\nu.$$

M and

$$\frac{1}{\pi}\int \frac{M(\lambda-l,\, \mu-m,\, \nu-n)dl\, dm\, dn}{l^2+m^2+n^2}$$

can thus be regarded as Fourier transforms of ϕ and ϕ/R.

Using then in the Schrödinger equation (1,1) we deduce the following integral equation

$$(l^2+m^2+n^2-k^2)M = \lambda \int \frac{M(l-x,\, m-y,\, n-z)}{x^2+y^2+z^2}\, dx\, dy\, dz, \qquad \ldots \quad \text{(A)}$$

where

$$k^2 = 2mE/h^2. \quad \text{and} \quad \lambda = 2me^2/\pi h^2.$$

M is here assumed to be finite and single-valued throughout the domain of integration, as also

$$\int |M|^2\, dx\, dy\, dz = 1.$$

(A) is the characteristic integral equation for M-functions of hydrogen; for $k^2 < 0$, it leads to the discrete spectrum, while $k^2 > 0$: yields the continuous spectrum.

It is easy to transform (A) to the Fredholm type, when $k^2 < 0 = -a^2$. We first transform the origin, *i.e.*, put $l-x = x'$, etc, in the right side. (A) becomes

$$(l^2+m^2+n^2-k^2)M = -\int \frac{\lambda M(x'y'z')dx'\, dy'\, dz'}{[(l-x')^2+(m-y')^2+(n-z')^2]} \; ; \qquad \ldots \quad (2{,}32)$$

when $k^2 = -a^2$, we make a similarity transformation

$$l = l'a, \; x' = ax'', \; etc.,$$

and put

$$M(l, m, n) = M'(l', m', n')$$

whereby

$$dx'\, dy'\, dz' = a^3 dx''\, dy''\, dz''$$

and

$$(l^2+m^2+n^2+a^2) = a^2(l'^2+m'^2+n'^2+1), \text{ etc.}$$

The relation becomes

$$(l^2+m^2+n^2+1)M = -\frac{\lambda}{a}\int \frac{M'(x'y'z')dx'dy'dz'}{(l-x')^2+(m-y')^2+(n-z')^2.}$$

If further

$$\sqrt{(1+l^2+m^2+n^2)}M = \phi,$$

then

$$\phi(A) = \kappa \int \frac{\phi(P)dv_P}{[(l-x)^2+(m-y)^2+(n-z)^2]\sqrt{(1+l^2+m^2+n^2)}\sqrt{(1+x^2+y^2+z^2)}}\,, \qquad \ldots \quad (2{,}33)$$

where

$$\kappa = \frac{e^2}{\pi h}\left(\frac{2m}{-E}\right)^{\frac{1}{2}}.$$

(2,33) is thus seen of the standard form

$$\phi(A) = -\lambda \int \phi(P)K(A, P)dv_p,$$

where the kernel is symmetrical in A and P. Such a transformation will however make the kernel imaginary within the domain, if $k^2 > 0$; to have a uniform treatment to cover both the cases, we will not utilise the transformation mentioned above.

III

We require the following simple result in our subsequent calculations. If r_1 and r_2 are the distances of a point $P(x, y, z)$ from two fixed points $A(a, b, c)$ and $F(f, g, h)$, the integral

$$I = \int \frac{dx\,dy\,dz}{r_1^2 r_2^2}$$

extended over the whole domain easily transforms to

$$\frac{2\pi}{c}\int_1^\infty \int_{-1}^1 \frac{dX dY}{X^2-Y^2}$$

in bipolar co-ordinates with $AF = 2c$,

$$dx\,dy\,dz = c^3(\cosh^2\xi - \cos^2\eta)\sinh\xi \sin\eta \; d\xi d\eta d\phi,$$

and

$$X = \cosh\xi \quad \text{and} \quad Y = \cos\eta,$$

whence, by easy integration,

$$I = \frac{\pi^3}{2c} = \frac{\pi^3}{AF}. \qquad \dots \quad (3,1)$$

Again, if

$$A(a, b, c) = \lambda \int \frac{M(a-x, b-y, c-z)dx\,dy\,dz}{x^2+y^2+z^2} = -\lambda \int \frac{M(x'y'z')dx'dy'dz'}{(a-x')^2+(b-y')^2+(c-z')^2} \qquad \dots \quad (3,2)$$

we have, by multiplying the equation with

$$\frac{da\,db\,dc}{(f-a)^2+(g-b)^2+(h-c)^2}$$

and integrating over the whole domain,

$$\int \frac{A(a\;b\;c)da\,db\,dc}{[(f-a)^2+(g-b)^2+(h-c)^2]} = -\lambda \int \int \frac{M(x'y'z')dx'\,dy'dz'\,da\,db\,dc}{[(a-x')^2+\dots][(f-a)^2+\dots]}.$$

Changing the order of integration and utilising (3,1) we have

$$\int \frac{M(x'y'z')dx'dy'dz'}{[(f-x')^2+(g-y')^2+(h-z')^2]^{\frac{1}{2}}} = -\frac{1}{\pi^3\lambda}\int \frac{A(abc)da\,db\,dc}{[(f-a)^2+(g-b)^2+(h-c)^2]}$$

$$= \frac{1}{\pi^3\lambda}\int \frac{A(f-a',\ldots)da'db'dc'}{a'^2+b'^2+c'^2}\ ;$$

hence, operating with $\nabla^2_{(fgh)}$ on both sides, as

$$\nabla^2_{(fgh)}\int \frac{M(x'y'z')dx'dy'dz'}{[(f-x')^2+\ldots]^{\frac{1}{2}}} = -4\pi M(f,g,h)$$

from potential theory, and

$$\nabla^2_{(fgh)}\{A(f-a,g-b,h-c)\} = (\nabla^2 A)(f-a,g-b,h-c),$$

we have finally

$$M(f,g,h) = -\frac{1}{4\pi^4\lambda}\int \frac{(\nabla^2 A)(f-a,\,g-b,\,h-c)\,da\,db\,dc}{a^2+b^2+c^2}$$

$$= \frac{1}{4\pi^4\lambda}\int \frac{(\nabla^2 A')dx'dy'dz'}{[(f-x')^2+(g-y')^2+(h-z')^2]} \qquad \ldots \quad (3,3)$$

as a solution of (3,2).

This important result furnishing a solution of integral equation (3,2) with obvious restrictions about the nature of the function A, enables us to tackle our present problem.

IV

According to a well-known theorem due to Hobson, if an operator $S\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)$ is constructed by substituting $\frac{d}{dx}$, ..., etc. for x, y, z in a solid harmonic S_n of degree n (a positive integer),

$$S_n\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)F(r^2) = 2^n S_n(x,y,z)\left(\frac{d}{dr^2}\right)^n F. \qquad \ldots \quad (4,1)$$

Let us assume, with regard to the integral equation (A), that

$$M(x,y,z) = S_n(x,y,z)f(r)$$

or

$$M(x,y,z) = S_n\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)A(r^2) \qquad \ldots \quad (4,2)$$

in view of (4,1); also $(r^2-k^2)M(x,y,z)$ can be written as

$$S_n\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)B(r^2) \qquad \dots \quad (4{,}3)$$

with the same operàtor, as the same surface-harmonic will occur in both. Also as

$$S_n\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)A = 2^n S_n(x,y,z)\left(\frac{d}{dr^2}\right)^n A = M,$$

and

$$S_n\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)B = 2^n S_n(x,y,z)\left(\frac{d}{dr^2}\right)^n B = (r^2-k^2)M,$$

we have

$$\left(\frac{d}{dr^2}\right)^n B = (r^2-k^2)\left(\frac{d}{dr^2}\right)^n A = \left(\frac{d}{dr^2}\right)^{n-1}[A(r^2-k^2)-nA],$$

implying

$$\frac{dB}{dr} = \frac{d}{dr}[A(r^2-k^2)]-2nAr. \qquad \dots \quad (4{,}4)$$

We have also from (A)

$$S_n\left[\frac{d}{da},\ \cdots\right]B = \lambda\int\frac{S_n\left[\frac{d}{d(a-x)},\ \cdots\right]A\,dx\,dy\,dz}{x^2+y^2+z^2}$$

$$= \lambda S_n\left(\frac{d}{da},\ \cdots\right)\int\frac{A[(a-x)^2+\dots]dx\,dy\,dz}{x^2+y^2+z^2}.$$

Hence removing the operator S_n

$$B = \lambda\int\frac{A[a-x)^2+\dots]dx\,dy\,dz,}{x^2+y^2+z^2}, \qquad \dots \quad (4{,}51)$$

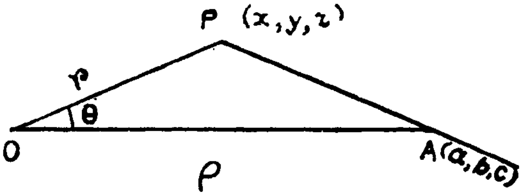
and hence on account of (3.3)

$$A = -\frac{1}{4\pi^4\lambda}\int\frac{(\nabla^2 B)(a-x,\,b-y,\,c-z)\,dx\,dy\,dz}{x^2+y^2+z^2}. \qquad \dots \quad (4{,}52)$$

We now perform the integration ; assuming

$$(a-x)^2+(b-y)^2+(c-z)^2 = f^2 = r^2+\rho^2-2r\rho\cos\theta$$

according to the accompanying figure



we have, as $fdf = r\rho \sin\theta$,

$$\rho B(\rho^2) = 2\pi\lambda \int_0^\infty \frac{dr}{r} \int_{f^2=(\rho-r)^2}^{f^2=(\rho+r)^2} \tfrac{1}{2} A\,(f^2)\, df^2$$

$$= 2\pi\lambda \int_0^\infty \frac{dx}{x}[G_1(\rho+x)-G_1(\rho-x)],$$

where

$$\frac{1}{x}\,\frac{dG}{dx} = A(x^2)$$

and G_1, an even function of x. Similarly from (4,52)

$$\rho A = -\frac{1}{2\pi^3\lambda}\int_0^\infty \frac{dx}{x}\,[L(\rho+x)-L(\rho-x)],$$

where

$$\frac{dL}{xdx} = \nabla^2 B(x^2);$$

hence integrating and putting

$$\frac{dQ}{dx} = L, \text{ or } \frac{1}{x}\,\frac{d^2Q}{dx^2} = \nabla^2 B$$

we have

$$G_1(\rho) = -\frac{1}{2\pi^3\lambda}\int \frac{dx}{x}\,[Q(\rho+x)-Q(\rho-x)],$$

where

$$\frac{1}{x}\,\frac{d^2Q}{dx^2} = \nabla^2 B = \frac{1}{x}\,\frac{d^2}{dx^2}\,(xB),$$

$$\text{or}\quad Q = xB,$$

and therefore an odd function of x.

Putting finally

$$\frac{xB(x^2)}{2\pi^2\lambda} = G_2 = \frac{Q}{2\pi^2\lambda},$$

we have the following relations between G_1 and G_2 in the skew-reciprocal form :

$$G_2(\rho) = \frac{1}{\pi}\int_0^\infty \frac{G_1(\rho+x)-G_1(\rho-x)}{x}\,dx, \qquad \ldots\ (4,53)$$

$$G_1(\rho) = -\frac{1}{\pi}\int_0^\infty \frac{G_2(\rho+x)-G_2(\rho-x)}{x}\,dx. \qquad \ldots\ (4,54)$$

If the functions G_1 and G_2 are introduced in (4,4) we have

$$\frac{d}{dr}\left[\frac{G_2}{r}\right] = \frac{1}{2\pi^2\lambda}\left[\frac{d}{dr}\left(\frac{r^2-k^2}{r}\cdot\frac{dG_1}{dr}\right)-2n\,\frac{dG_1}{dr}\right],$$

or

$$G_2 = \frac{1}{2\pi^2\lambda}\left[(x^2-k^2)\frac{dG_1}{dx} - 2nxG_1\right],$$

or

$$\frac{G_2}{(x^2-k^2)^{n+1}} = \frac{1}{2\pi^2\lambda}\frac{d}{dx}\left[\frac{G_1}{(x^2-k^2)^n}\right]. \qquad \ldots \quad (4{,}55)$$

The skew-reciprocal relations (4,53) and (4,54) then at once suggest that

$$\frac{G_1}{(x^2-k^2)^{n+1}} = -\frac{1}{2\pi^2\lambda}\frac{d}{dx}\left[\frac{G_2}{(x^2-k^2)^n}\right] \qquad \ldots \quad (4{,}56)$$

is true at the same time.

(4,55) and (4,56), lead at once, by integration, to

$$G_1 = \mathbf{A}(x^2-k^2)^n \cos\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{x-k}{x+k}\right|\right),$$

$$G_2 = -\mathbf{A}(x^2-k^2)^n \sin\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{x-k}{x+k}\right|\right), \qquad (4{,}57)$$

when $k^2 > 0$; when $k^2 = -a^2$, on the other hand,

$$G_1 = \mathbf{B}\,(x^2+a^2)^n \cos\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{x}{a}\right),$$

$$G_2 = -\mathbf{B}(x^2+a^2)^n \sin\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{x}{a}\right). \qquad \ldots \quad (4{,}58)$$

The two equivalent forms of M are

$$M(x, y, z) = \mathbf{A}\; S_n(x, y, z)\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}\left[(r^2-k^2)^n \cos\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{r-k}{r+k}\right|\right)\right]$$

or

$$= -2\pi^2\lambda\,\frac{\mathbf{A}S_n(x, y, z)}{r^2-k^2}\left(\frac{1}{r}\frac{d}{dr}\right)^n\left[\frac{(r^2-k^2)^n}{r}\sin\left(\frac{2\pi^2\lambda}{2k}\log\left|\frac{r-k}{r+k}\right|\right)\right] \qquad \ldots \quad (4{,}591)$$

for the case : $k^2 > 0$; while, if $k^2 = -a^2$,

$$M(x, y, z) = \mathbf{B}S_n(x, y, z)\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}\left[(r^2+a^2)^n \cos\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{r}{a}\right)\right]$$

or

$$= -2\pi^2\lambda\,\mathbf{B}\,\frac{S_n(x, y, z)}{r^2+a^2}\left(\frac{1}{r}\frac{d}{dr}\right)^n\left[\frac{(r^2+a^2)^n}{r}\sin\left(\frac{2\pi^2\lambda}{r}\tan^{-1}\frac{r}{a}\right)\right]. \qquad \ldots \quad (4.592)$$

V

As M is to be single-valued, we see, in the case of $k^2\ (< 0) = -a^2$, that a restriction comes in regarding the choice of constants of the problem.

$\tan^{-1}(r/a)$ is a multiple-valued function of the form $\theta + N\pi$; if therefore $\cos\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{r}{a}\right)$ and $\sin\left(\frac{2\pi^2\lambda}{a}\tan^{-1}\frac{r}{a}\right)$ are single-valued, it follows that $\pi^2\lambda/a$ must be an integer $= N$. This gives discrete energy-values :

$$k^2 = \frac{2mE}{h^2} = -\frac{\pi^4\lambda^2}{N^2},$$

or

$$E = -\frac{2\pi^2 e^4 m}{h^2 N^2}$$

agreeing with the well-known result. Also as

$$2\tan^{-1}\frac{r}{a} = \cos^{-1}\frac{a^2-r^2}{a^2+r^2},$$

$$\cos\left(N\cos^{-1}\frac{a^2-r^2}{a^2+r^2}\right) = \frac{f_N(r^2)}{(a^2+r^2)^N}$$

we see that $M = 0$ unless $N \geqslant n+1$.

Again starting from the series

$$\frac{1}{1-2h\cos\theta+h^2} = \sum_1 h^{N-1}\frac{\sin N\theta}{\sin\theta}, \quad \frac{1-h\cos\theta}{1-2h\cos\theta+h^2} = \sum_0 h^N \cos N\theta, \qquad (5,1)$$

and writing

$$\cos\theta = \frac{a^2-r^2}{a^2+r^2} \text{ and } \sin\theta = \frac{2ar}{a^2+r^2}$$

we have

$$\frac{2a(a^2+r^2)}{\{(1-h)^2a^2+(1+h)^2r^2\}} = \Sigma\, h^{N-1}\,\frac{(a^2+r^2)}{r}\,\sin N\theta, \qquad \ldots\ (5,2)$$

$$\frac{a^2(1-h)+r^2(1+h)}{\{(1-h)^2a^2+(1+h)^2r^2\}} = \Sigma h^N \cos N\theta. \qquad \ldots\ (5,3)$$

Multiplying (5,2) with $(a^2+r^2)^{n-1}$ and differentiating n times with regard to r, we have

$$\frac{1}{[(1-h)^2a^2+(1+h)^2r^2]^{n+1}}$$

$$= \frac{(-1)^n}{n!2^{2n+1}\,a^{2n+1}} \Sigma\, h^{N-n-1} \left(\frac{d}{dr^2} \right)^n \left[\frac{(a^2+r^2)^n}{r} \sin \left(N \cos^{-1} \left(\frac{a^2-r^2}{a^2+r^2} \right) \right] ; \quad \ldots \quad (5,4)$$

from (5,3) we deduce similarly

$$\left(\frac{1-h^2}{[(1-h)^2a^2+(1+h)^2r^2]^{n+2}} \right)$$

$$= \frac{(-1)^{n+1}}{2^{2n+1}a^{2n+2}} \cdot \frac{1}{(n+1)!} \Sigma\, h^{N-n-1} \left(\frac{d}{dr^2} \right)^{n+1} \left[(a^2+r^2)^n \cos \left(N \cos^{-1} \frac{a^2-r^2}{a^2+r^2} \right) \right] . \quad (5,5)$$

Remembering the Gegenbauer expansion

$$\frac{1}{(\alpha^2-2\alpha h \cos\theta+h^2)^\nu} = \frac{1}{\alpha^{2\nu}} + \sum_\mu \frac{(2h)^\mu}{\alpha^{2\nu+\mu}} F^\nu_\mu (\cos\theta)$$

we can easily deduce, if $\nu = n+1$, $\alpha = 1$ and

$$\cos\theta = \frac{a^2-r^2}{a^2+r^2},$$

the following results :

$$\frac{1}{[a^2(1-h)^2+r^2(1+h)^2]^{n+1}} = \frac{1}{(a^2+r^2)^{n+1}} \left[1+\sum_1 (2h)^\lambda F^{n+1}_\lambda \left(\frac{a^2-r^2}{a^2+r^2} \right) \right], \quad \ldots \quad (5,6)$$

$$\frac{1-h^2}{[a^2(1-h)^2+r^2(1+h)^2]^{n+2}} = \frac{1}{(a^2+r^2)^{n+2}} \left[1+\sum_\lambda \frac{(n+1+\lambda)}{n+1} (2h)^\lambda F^{n+1}_\lambda \left(\frac{a^2-r^2}{a^2+r^2} \right) \right] . \quad (5,7)$$

Comparing (5.4) and (5.5) with (5.6) and (5.7) we have

$$\frac{1}{(a^2+r^2)} \left(\frac{d}{dr^2} \right)^n \left[\frac{(a^2+r^2)^n}{r} \sin N\theta \right] = \frac{(-1)^n 2^{N+n} a^{2n+1} n!}{(a^2+r^2)^{n+2}} F^{n+1}_{N-n-1} \left(\frac{a^2-r^2}{a^2+r^2} \right),$$

and

$$\left(\frac{d}{dr^2} \right)^{n+1} \left[(a^2+r^2)^n \cos N\theta \right] = \frac{(-1)^{n+1} 2^{N+n} a^{2n+2} N \cdot n!}{(a^2+r^2)^{n+2}} F^{n+1}_{N-n-1} \left(\frac{a^2-r^2}{a^2+r^2} \right).$$

Hence

$$M(x, y, z) = \mathbf{B} S_n(x, y, z) \left(\frac{1}{r} \frac{d}{dr} \right)^{n+1} \left[(r^2+a^2)^n \cos \left(N \cos^{-1} \frac{a^2-r^2}{a^2+r^2} \right) \right]$$

or

$$= -2Na\,\frac{\mathbf{B}S_n(x,y,z)}{a^2+r^2}\left(\frac{d}{rdr}\right)^n\left[\frac{(a^2+r^2)^n}{r}\sin\left(N\cos^{-1}\frac{a^2-r^2}{a^2+r^2}\right)\right]$$

$$= CS_n(x,y,z)\,F^{n+1}_{N-n-1}\left(\frac{a^2-r^2}{a^2+r^2}\right)$$

VI

When $k^2 < 0$, the problem can be transformed to a homogeneous integral equation of the Fredholm type, with a symmetrical kernel

$$\phi(A) = \lambda\int \phi(P)K(A,P)dv_P.$$

The results that we have obtained above enable us also to say that

$$\phi = C\sqrt{1+x^2+y^2+z^2}\;S_n(x,y,z)\left(\frac{1}{r}\frac{d}{dr}\right)^{n+1}\left[(1+x^2+y^2+z^2)^n\cos\left(N\cos^{-1}\frac{1-r^2}{1+r^2}\right)\right]$$

$$= D\frac{S_n(x,y,z)}{(1+x^2+y^2+z^2)^{n+3/2}}F^{n+1}_\lambda\left(\frac{1-r^2}{1+r^2}\right) \qquad \ldots \quad (6,1)$$

are eigen-functions corresponding to the eigen-value $-(n+1+\lambda)/\pi^2$ for the kernel

$$K(A,P) = \frac{1}{(x-x')^2+(y-y')^2+(z-z')^2\sqrt{(1+x^2+y^2+z^2)}\sqrt{(1+x'^2+y'^2+z'^2)}}.$$

Writing

$$M(x,y,z) = \frac{\mathbf{C}S_n(x,y,z)}{(a^2+r^2)^{n+2}}F^{n+1}_\lambda\left(\frac{a^2-r^2}{a^2+r^2}\right) = \mathbf{C}\,Y_n(\theta,\phi)\frac{r^n}{(a^2+r^2)^{n+2}}F^{n+1}_\lambda\left(\frac{a^2-r^2}{a^2+r^2}\right);$$

also

$$\int M^2dxdydz = 1 = \mathbf{C}^2\int Y_n^2(\theta,\phi)d\omega\int\frac{r^{2n+2}}{(a^2+r^2)^{2n+4}}\left[F^{n+1}_\lambda\left(\frac{a^2-r^2}{a^2+r^2}\right)\right]^2dr.$$

To determine the radial integral, we use (5,7) and write

$$\mathrm{I} = \int\frac{(1-h^2)(1-t^2)r^{2n+2}dr}{[a^2(1-h)^2+r^2(1+h)^2]^{n+2}[a^2(1-t)^2+r^2(1+t)^2]^{n+2}}$$

$$= \sum_\lambda\sum_\sigma\frac{(n+1+\lambda)(n+1+\sigma)}{(n+1)^2}(2h)^\lambda\,(2t)^\sigma\int\int\frac{F^{n+1}_\lambda F^{n+1}_\sigma r^{2n+2}}{(a^2+r^2)^{2n+4}}dr,$$

which reduces to

$$I = \frac{(-1)^{n+1}}{(n+1)!}\cdot\frac{(1-h^2)(1-t^2)}{(1+h)^{2n+4}(1+t)^{2n+4}}\cdot\left\{\frac{d}{d(p+q)}\right\}^{n+1}\int_0^\infty\frac{dr}{[p+r^2][q+r^2]},$$

where

$$p = \frac{a^2(1-h)^2}{(1+h)^2},\ q = \frac{a^2(1-t)^2}{(1+t)^2},$$

when, by easy calculation,

$$I = \frac{\pi}{2^{4n+5}a^{2n+5}}\cdot\frac{(2n+1)\,!}{n!(n+1)!}\,\frac{(1+h)(1+t)}{(1-ht)^{2n+3}}\,. \qquad \dots \quad (6.2)$$

Equating coefficient of $(ht)^\lambda$

$$\int \frac{r^{2n+2}}{(a^2+r^2)^{2n+4}}\left(F_\lambda^{n+1}\right)^2 dr = \frac{\pi(2n+1+\lambda)!}{2^{4n+2\lambda+5}\,a^{2n+5}(n+1+\lambda)\lambda!\{n!\}^2}\,;$$

hence

$$\overline{M}\ (\text{normalised}) = 2^{2n+\lambda+3}a^{n+3}\left(\frac{N}{2\pi a}\right)^{\frac{1}{2}} n\,!\left\{\frac{\lambda\,!}{(N+n)!}\right\}^{\frac{1}{2}}\frac{F_\lambda^{n+1}\left(\dfrac{a^2-r^2}{a^2+r^2}\right)}{(a^2+r^2)^{n+2}}\overline{Y}_n(\theta,\phi), \qquad \dots \quad (6.3]$$

where $\overline{Y}_n(\theta, \phi)$ is taken to be a normalised spherical harmonic.

We can normalise the eigen-functions of the homogeneous integral equation, for which similarly the value of

$$\int \frac{r^{2n+2}}{(1+r^2)^{2n+3}}\left(F_\lambda^{n+1}\right)^2 dr$$

will be necessary; for it, we utilise again (5.7) and write

$$\mathrm{I} = \int_0^\infty \frac{(1-h^2]r^{2n+2}dr}{[(1-h)^2+(1+h)^2r^2]^{n+2}[(1-t)^2+(1+t)^2r^2]^{n+1}}$$

$$= \sum_\lambda \sum_\sigma (2h)^\lambda(2t)^\sigma\,\frac{(n+1+\lambda)}{(n+1)}\int\frac{F_\lambda^{n+1}\,F_\sigma^{n+1}\,r^{2n+2}\,dr}{(1+r^2)^{2n+3}}\,;$$

after easy integration

$$I = \frac{\pi}{2^{4n+4}}\cdot\frac{(2n+1)!}{n!\,(n+1)\,!}\cdot\cdot\frac{1}{(1-ht)^{2n+2}}\,. \qquad \dots \quad (6.4)$$

Equating coefficient we have

$$\int\frac{\{F_\lambda^{n+1}\}^2\,r^{2n+2}\,dr}{(1+r^2)^{2n+3}} = \frac{\pi}{2^{4n+2\lambda+4}(n!)^2(n+1+\lambda)}\cdot\frac{(2n+1+\lambda)!}{\lambda!}.$$

The normalised eigen-functions of the homogeneous equation is

$$\psi = \frac{2^{2n+\lambda+2}n!\,(n+1+\lambda)^{\frac{1}{2}}}{\cdot\ \sqrt{\pi}}\cdot\left(\frac{\lambda!}{(2n+1+\lambda)\,!}\right)^{\frac{1}{2}}\frac{Y_n(\theta,\phi)r^n}{(1+r^2)^{n+3/2}}\,F_\lambda^{n+1}\left(\frac{1-r^2}{1+r^2}\right). \dots \ (6.5)$$

The completeness of the eigen-functions series will mean the equality

$$K(x, y) = \sum_{n} \frac{\pi^2 \psi_r^n(x)\, \psi_r^n(y)}{(n+1+\lambda)}$$

(multiple summations as there are repeated roots) leading to the relation

$$\frac{1}{[(x-x')^2+(y-y')^2+(z-z')^2][1+r^2]^{\frac{1}{2}}[1+r'^2]^{\frac{1}{2}}}$$

$$= \sum_{1} \frac{\pi^2}{N} \sum_{n=0}^{n=N} \frac{2^{2N+2n+2}}{\pi} \quad (n!)^2\, N \left[\frac{(N-n-1)!}{(N+n)!} \right] \frac{r^n}{(1+r^2)^{3/2}} \cdot \frac{r'^n}{(1+r'^2)^{3/2}} \; F_\lambda^{n+1} \; F'^{n+1}_\lambda$$

$$\times \frac{(2n+1)}{4\pi} \left[\; \sum_{1}^{n} 2P_n^m(\mu) P_n^m(\mu') \; \frac{(n-m)!}{(n+m)!} \cos m(\phi-\phi') + P_n(\mu)\, P_n(\mu') \right]$$

and leading to an expansion of

$$\frac{1}{(x-x')^2+(y-y')^2+(z-z')^2} = \sum_{N} 2^{2N} \sum_{0}^{N-1} 2^{2n} F_{N-n-1}^{n+1} \left(\frac{1-r^2}{1+r^2} \right) F_{N-n-1}^{n+1}$$

$$\left(\frac{1-r'^2}{1+r'^2} \right) \frac{r^n\, r'^n}{(1+r^2)^{n+1}(1+r'^2)^{n+1}} \quad \left[\frac{(N-n-1!)}{(N+n)!} \right] (n!)^2 \left[P_n(\mu) P_n(\mu') \right.$$

$$\left. +2 \sum_{1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\mu) P_n^m(\mu') \cos m(\phi-\phi') \right]; \qquad \ldots \quad (6.7)$$

by a change of axis making $(x'\; y'\; z')$ on the z axis

$$\frac{1}{r^2+r'^2-2rr'\mu} = \sum_{N} 2^{2N} \sum_{n} 2^{2n} \; F_{N-n-1}^{n+1} \left(\frac{1-r^2}{1+r^2} \right) \cdot F_{N-n-1}^{n+1} \left(\frac{1-r'^2}{1+r'^2} \right) \frac{r^n r'^n}{(1+r^2)^{n+1}\,(1+r'^2)^{n+1}}$$

$$\times (n!)^2 \left[\frac{(N-n-1)\,!}{(N+n)\,!} \right] (2n+1) \;\; P_n(\mu) \qquad (6.8)$$

from the well-known addition theorem of $P_n(\mu)$.

To verify the correctness of the result we note that

$$\frac{2}{2[r^2+r'^2-2rr'\mu]}=\frac{2}{(1+r^2)(r'^2+1)\left[\;1-\dfrac{1-r^2}{1+r^2}\cdot\dfrac{1-r'^2}{1+r'^2}-\dfrac{4rr'}{(1+r^2)(1+r'^2)}\cos\phi\;\right]}\;;$$

hence writing

$$\cos\,\theta=\frac{1-r^2}{1+r^2},\quad \cos\theta'=\frac{1-r'^2}{1+r'^2},$$

we have

$$\frac{1}{r+r'^2-2rr'\mu^2}=\frac{2}{(1+r^2)(1+r'^2)[1-\cos\theta\,\cos\theta'-\sin\theta\,\sin\theta'\,\cos\phi]}$$

$$=\frac{4}{(1+r^2)(1+r'^2)}\sum_{1}^{N}\frac{\sin N\chi}{\sin\chi},\tag{6·81}$$

if $\pi/2>\chi>0$ from (5.1), and $\cos\chi=\cos\theta\,\cos\theta'+\sin\theta\,\sin\theta'\,\cos\phi$; hence

$$\frac{\sin N\chi}{\sin\chi}=\sum_{n=0}^{N-1}\;2^{2(N-1)}\,(2n+1)(n!)^2\left[\;\frac{(N-n-1)\,!}{(N+n)\,!}\;\right]$$

$$\times\;\sin^n\theta\,\sin^n\theta' F^{n+1}_{N-n-1}\,(\cos\theta)\cdot F^{n+1}_{N-n-1}\,(\cos\theta')\cdot P_n(\cos\phi),\tag{6.9}$$

a result which is a particular case of a general class of identity deduced by Gegenbauer (Whittaker, 1927).

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Letters to the Editor

GERMANIUM IN SPHALERITE FROM NEPAL

IN VIEW OF THE RECENT USE OF GERMANIUM crystal as a rectifier for ultra-short radio waves, it is worth while to make a thorough investigation of the possible sources of germanium in India. Recently, one of us made a search for germanium in coal ashes from different coalfields in India and a fair concentration of germanium was detected in a few samples[1,2].

Argyrodite, canfieldite and germanite are the only ores rich in germanium known at present. But none of them have been reported to occur in India. Zinc blende from different countries has been investigated by different workers[3] and in majority of samples germanium has been detected in traces.

Specimens of sphalerite collected from Nepal by one us have been spectroscopically examined. Two of these samples have been found to contain germanium (FIGS. 1-4). The following metals have also been detected in the mineral : Sn, Pb, Mg, Al, Si, Ga, Cd, Ge, Ag and Fe.

During our investigation, we found that germanium occurs concentrated in the magnetic portion of the mineral which seems to be almost pure iron oxide with traces of sulphur, lead, zinc, gallium and germanium. The non-magnetic portions were also examined spectroscopically and found to contain germanium only in traces.

Germanium was estimated colorimetrically according to the method by Marcel Orliac[4] slightly modified by us.

The percentage of germanium in the magnetic portion has been found to be about 0·24 while it varies in the non-magnetic portion from 0·0072 to 0·04. 300 mg. of pure GeO_2 has been extracted from 160 gm. of the magnetic portion of the mineral.

The most abundant source of germanium at present in U.S.A. is the concentrate from residues obtained during the melting of zinc ores. Buchanan reports as much as 0·25 per cent of GeO_2 from these residues. As the magnetic fraction from sphalerite contains about 0·24 per cent of germanium, it can be regarded as a good source of germanium.

It appears that germanium has not been detected in any iron ore so far ; thus this new mineral needs further study. We have spectroscopically examined a few haematite and pyrites samples but none of them contains germanium.

We take this opportunity to thank the Nepal Government and the Director of the Nepal Bureau of Mines for the samples of sphalerite. Our grateful thanks are due

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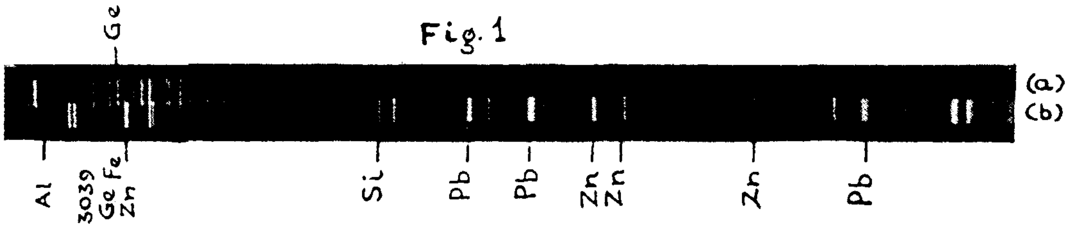
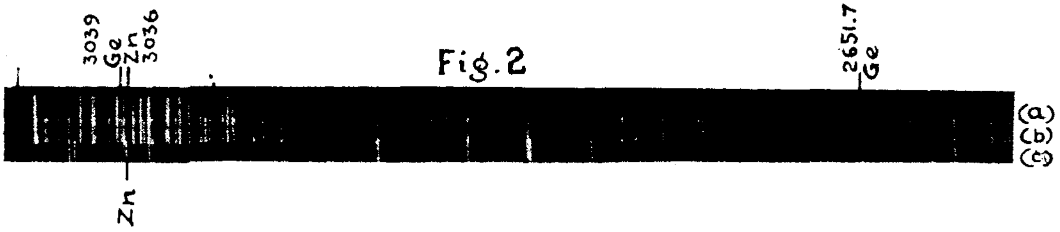
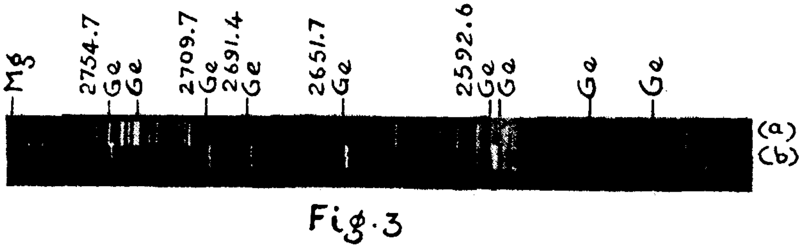
The first reference on p. 271 makes a wrong ascription to Dutta and Sen instead of Dutta and Bose, in the original publication itself.







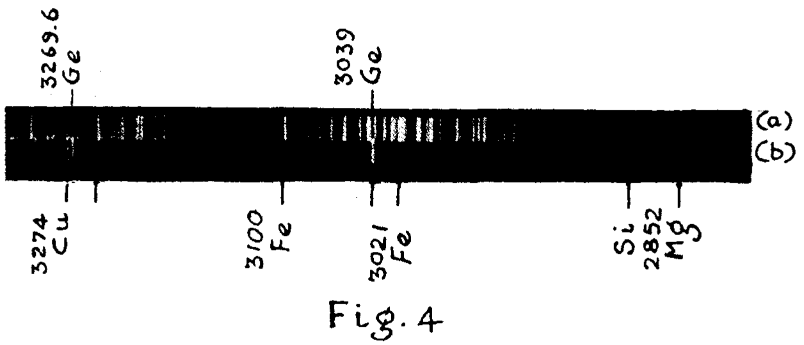


FIG. 1 — (a) FE, (b) SPHALERITE. FIG. 2 — (a) FE, (b) MAGNETIC SAMPLE, (c) NON-MAGNETIC SAMPLE. FIG. 3 — (a) FE, (b) GE WITH TRACE OF FE. FIG. 4 — (a) FE, (b) GE WITH TRACE OF FE.

to Prof. P. B. Sarkar for his keen interest during the progress of the work. One of us (R.K.D.) is indebted to the *Council of Scientific & Industrial Research* for a research grant.

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R. K. DATTA

University College of
Science & Technology
Calcutta
October 27, 1949

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Reprinted from *J Sci Indust Res* **9B**, pp. 52-53, 1950 (Council of Scientific & Industrial Research, New Delhi).

Extraction of Germanium from Sphalerite Collected from Nepal—Part I

R. K. DUTTA & S. N. BOSE
University College of Science, Calcutta

A method for the extraction of germanium from the magnetic fraction of the mineral sphalerite has been described. The fraction which is shown to be magnetite contains 0·24 per cent germanium.

IN a previous communication[1], we described a magnetic fraction of the mineral sphalerite collected from Nepal. On chemical analysis, this fraction was found to be mainly ferric oxide with a coating of ferrous sulphide. It contains germanium to the extent of 0·24 per cent. That the mineral is magnetite has also been confirmed by X-ray analysis. Coal ash, euxenite and some specimens of zinc blende are known to contain germanium in small quantities, but its association with magnetite has not been reported.

Experimental

Analysis of the Mineral — The spectrogram revealed the existence of iron, silicon, lead (trace) and germanium.

Total Iron — A weighed amount of finely powdered mineral was dissolved in 1 : 1 hydrochloric acid and the total iron estimated by titration with potassium dichromate after reduction with stannous chloride.

Silica was estimated by the usual method, and the value verified by determining the loss in weight by treatment with hydrofluoric acid.

Sulphur — The mineral was decomposed with sodium peroxide and the sulphur estimated as barium sulphate.

Ferrous iron was estimated by decomposing the mineral with 5 c.c. of concentrated hydrochloric acid and 10 c.c. of hydrofluoric acid in an atmosphere of carbon dioxide, and subsequent titration with potassium dichromate using diphenylamine sulphonate as indicator.

Germanium — Of the various methods known for the estimation of germanium[3-5], Orliac's colorimetric method[5] with slight modification was found to be the most convenient. A weighed amount of the mineral was carefully roasted at 500°C. in an electric furnace and transferred to an all-glass distillation apparatus. 40 c.c. of 1 : 1 hydrochloric acid were added and the flask gently heated in a current of air. The distillate was collected in a flask containing dilute caustic soda solution and a drop of phenolphthalein indicator solution. After the distillation was complete, the solution was acidified with dilute hydrochloric acid and the volume made up to 100 c.c. To 10 c.c. of this solution, 1 c.c. ammonium molybdate solution (9 per cent) and 10 c.c. of alkaline potassium stannite solution containing 5 gm. of hydrated stannous chloride and 300 gm. of potassium hydroxide per litre were added and the total volume made up to 40 c.c. A series of standard solutions of germanium tetrachloride were

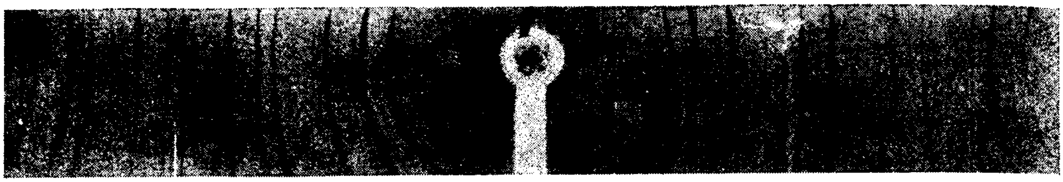

FIG. 1 — X-RAY POWDER DIAGRAM OF THE NON-MAGNETIC PORTION FROM THE MINERAL SPHALERITE.

prepared and the same volume of ammonium molybdate and alkaline stannite were added and the total volume in each case made up to 40 c.c. After an interval of 5 min. the transmittancies of the solutions were determined in a Lumetron photo-electric colorimeter, using 440 mμ filter, and a standard graph was constructed. From this graph germanium in the mineral was found to be 0·24 per cent.

The results of the analysis are tabulated below:

	%		
Total iron	69·5	Fe_2O_3	66·28
Ferrous iron	23·1	FeO	27·02
Sulphur	1·19	FeS	3·27
Silica	2·16	SiO_2	2·16
Germanium	0·24	GeO_2	0·34
			99·07

The mineral is apparently magnetite with a coating of ferrous sulphide. This has now been confirmed by X-ray powder diagram kindly supplied by Mons. Pierre Urbain, Director, Hydrological Institute, Sorbonne, Paris. The rontgenogram is reproduced in Fig. 1.

Extraction of Germanium Dioxide — None of the methods employed by Winkler for the extraction of germanium from Argyrodite was adopted, as the associated elements in this mineral are quite different from those in Argyrodite. After various trials the final procedure adopted was based upon the volatility of germanium tetrachloride and is an elaboration of the procedure described by Buchanan[2].

160 gm. of the finely powdered mineral were roasted in an electric muffle at about 550°C. for 12 hr. when all the sulphide was converted to sulphate and oxide. The roast ore was transferred to an all-glass Claisen flask. 400 c.c. of 1 : 1 hydrochloric acid were added and germanium tetrachloride distilled off in a current of hydrochloric acid gas till the volume is reduced to one-fourth. This distillation was repeated with further additions of hydrochloric acid. A rapid stream of hydrogen sulphide gas was then passed through the distillate for about 15 min. when a milky precipitate of germanium disulphide was obtained (for complete precipitation, the acid strength was maintained at 4N to 6N by the addition of concentrated hydrochloric acid). The mixture was set aside in a stoppered bottle for 48 hr., the sulphide separated by filtration and washed with 6N sulphuric acid saturated with hydrogen sulphide till it was free from chloride. The precipitate on the filter paper was treated with aqueous ammonia (about 8N) and the filtrate collected. The germanium disulphide was thus obtained as the thio-salt. The filtrate was evaporated, hydrogen peroxide added and again evaporated to dryness. The dried mass was carefully heated to remove ammonium sulphate completely, transferred to a crucible with water, evaporated and finally ignited to germanium dioxide at 600°C. The yield was 330 mg.

The spectrum of germanium dioxide thus obtained reveals the existence of traces of iron, lead, phosphorus (impurities from the hydrogen peroxide used).

Acknowledgement

Our grateful thanks are due to Prof. P. B. Sarkar for his keen interest in this investigation, and to the *Council of Scientific Industrial Research* for granting a scholarship to one of us (R. K. Dutta).

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Reprinted from *J Sci Indust Res* **9B**, pp. 251-252, 1950 (Council of Scientific & Industrial Research, New Delhi).

Extraction of Germanium from Sphalerite Collected from Nepal—Part II

R. K. DUTTA & S. N. BOSE

University College of Science, Calcutta

A method for the extraction of germanium from the non-magnetic fraction of the mineral sphalerite has been described.

IN the previous communication[1] we have described the extraction of germanium dioxide from the magnetic fraction (mainly magnetite) of sphalerite. The non-magnetic portion left over was found to contain traces (0·006-0·04 per cent) of germanium.

The spectrogram of this portion revealed the presence of the following elements: lead, zinc, iron, silicon, germanium, tin, copper, cadmium, silver and antimony, of which the first two predominate. The ore was analysed as before and the germanium estimated according to the method already described.

The analytical results are as follows:

	%
Pb	14·2
Zn	48·4
S	27·3
SiO_2	5·6
Fe	2·3
Ge	0·0021

From the analytical data, this fraction of the mineral appears to be mainly a sulphate of zinc and lead.

Extraction of Germanium

A procedure different from the one adopted for magnetite had to be adopted for the extraction of germanium from this fraction. When the ore is treated with sulphuric acid the insoluble lead sulphate formed retains the major portion of the germanium but if the sulphuric acid is completely fumed off, most of the germanium goes into solution and only traces remain over with the precipitates of lead sulphate and sulphur.

The methods of Winkler, Buchanan[2], etc., were found to be unsuitable for the extraction of germanium. Tchakirian's method[3] was finally adopted. The insoluble residue was refluxed with a saturated solution of ammonium bioxalate and, curiously enough, not even a trace of germanium was found in the solution. A new method based on the fact that the insoluble residue containing free sulphur is totally converted to lead sulphide on treatment with concentrated sodium hydroxide solution was adopted. The dismutation of free sulphur in strongly alkaline medium takes place as follows:

$$4S + 6O\bar{H} = 2S^{--} + \bar{S}_2\bar{O}_3 + 3H_2O$$

The lead sulphide so formed carried with it the whole of germanium as disulphide and the lead sulphide is separated by treatment with 2N hydrochloric acid.

Experimental

400 gm. of the ore were decomposed by an excess of 1 : 1 sulphuric acid, the resulting mixture heated to dryness and the excess acid fumed off. A large part of the free sulphur was removed by this treatment. The dried mass was extracted with hot water and filtered. Sodium sulphide, just sufficient to precipitate one-tenth of the total zinc, was added[4]. The filtrate was rejected as spectroscopic examination showed the absence of germanium. The residue was treated with 4N sulphuric acid and digested on a water bath for an hour and filtered. Germanium disulphide was left in the residue and most of the elements including zinc were thus removed. The residue was treated with sulphuric and nitric acids and dried. It was then distilled with 50 c.c. of 1 : 1 hydrochloric acid and the distillate treated according to the procedure already described. Germanium dioxide thus obtained was found from spectroscopic evidence to contain traces of antimony and tin.

When the residue was treated with concentrated sodium hydroxide solution, the whole of germanium was precipitated along with lead sulphide. This precipitate was digested with 2N hydrochloric acid, filtered and washed. Almost the whole of lead goes into solution leaving behind germanium disulphide, silicon dioxide, titanium, silver and traces of lead. Silicon dioxide was the principal constituent. All the vapours issuing during the treatment were collected and found to be spectroscopically free from germanium showing that germanium disulphide is not decomposed by 2N hydrochloric acid. The residue was then treated with sulphuric and hydrofluoric acids and evaporated on a water bath, when all the silica was removed. The mass was then distilled with hydrochloric acid, as previously described, and the germanium dioxide separated. The resulting product contained traces of phosphorus derived from hydrogen peroxide. The yield was 240 mg. of germanium dioxide from 4 kg. of the non-magnetic fraction.

Our best thanks are due to Prof. P. B. Sarkar for his keen interest in the investigation and to the *Council of Scientific & Industrial Research* for financial help to one of us (R. K. Dutta).

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PHYSIQUE THÉORIQUE. — *Les identités de divergence dans la nouvelle théorie unitaire.* Note de M. S. N. **Bose**, présentée par M. Louis de Broglie.

Einstein (¹) a déduit les quatre identités de divergence de sa nouvelle théorie des identités de Bianchi, en y levant l'ambiguïté de la différenciation covariante.

La méthode de Hilbert (²) légèrement modifiée fournit aussi immédiatement le résultat; il est peut-être intéressant d'indiquer le mode de calcul.

La relation entre les coefficients de connexion affine $\Gamma^{\alpha}_{\mu\nu}$ et $g^{\mu\nu}$ n'est pas simple dans la nouvelle théorie. On ne peut donc appliquer directement ici la méthode de Hilbert de variation d'une fonction invariante de $g^{\mu\nu}$ et de ses dérivées. Dans la nouvelle théorie, on ne suppose pas initialement cette relation entre Γ et g et la variation donne les équations ainsi que les équations de connexion de Γ et g. La nouvelle démonstration proposée tient compte de ceci. On comprendra la méthode d'après les indications suivantes.

Formons le tenseur $R_{\mu\nu}$, contracté du tenseur de Riemann généralisé. Son caractère covariant est évident d'après la loi de transformation de $\Gamma^{\alpha}_{\mu\nu}$ comme dans la théorie symétrique. $g^{\mu\nu} R_{\mu\nu}$ est donc un invariant dans les changements de coordonnées, de même que l'expression $|g|^{1/2} dx_1\, dx_2\, dx_3\, dx_4 = |g|^{1/2} dv_4$.

Si $R_{\mu\nu} = 0$ l'intégrale

$$I = \int R_{\mu\nu}\, \delta g^{\mu\nu} \sqrt{g}\, dv_4 = 0 \tag{1}$$

pour une variation quelconque des seuls $g^{\mu\nu}$; $R_{\mu\nu}$ étant fonction des coefficients de connexion affine n'est pas changé dans la variation.

Observons avec Hilbert (²) que

$$p^{\mu\nu} = \frac{\delta g^{\mu\nu}}{\delta x^{\sigma}} \varphi^{\sigma} - g^{\mu\sigma} \varphi^{\nu}_{\sigma} - g^{\sigma\nu} \varphi^{\mu}_{\sigma} \tag{2}$$

construit à partir d'un vecteur contrevariant arbitraire φ^{σ} peut être pris pour $\delta g^{\mu\nu}$ dans (1). D'où

$$0 = \int R_{\mu\nu} \sqrt{|g|} \left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}} \varphi^{\sigma} - g^{\mu\sigma} \varphi^{\nu}_{\sigma} - g^{\sigma\nu} \varphi^{\alpha}_{\sigma} \right) dv_4. \tag{3}$$

(¹) *Canad. J. Math.*, **2**, 1950, p. 120 et suiv.

(²) *Math. Ann.*, **92**, 1924, p. 1 et suiv.

L'intégration est étendue à une multiplicité à quatre dimensions limitée par une multiplicité à trois dimensions et l'on a partout $R_{\mu\nu} = 0$.

Nous observons que

$$(4) \qquad \frac{d}{dx^\sigma}[R_{\mu\nu} g^{\mu\nu} \varphi^\sigma \sqrt{|g|}] = \frac{\partial R_{\mu\nu}}{\partial x^\sigma} g^{\mu\nu} \sqrt{|g|} \varphi^\sigma + R_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x^\sigma} \sqrt{|g|} \varphi^\sigma + R_{\mu\nu} g^{\mu\nu} \left[\sqrt{g} \frac{\partial \varphi^\sigma}{\partial x^\sigma} + \frac{\partial \sqrt{g}}{\partial x^\sigma} \varphi^\sigma \right]$$

Si nous considérons $\varphi^\sigma \equiv \Delta x^\sigma$ comme une variation convenable petite et arbitraire des coordonnées, variation qui s'annule à la frontière, nous observons que

$$(5) \qquad \left. \begin{array}{l} \dfrac{\partial \phi^\sigma}{\partial x^\sigma} dv_4 = \Delta dv_4 \quad \text{et} \quad \dfrac{\partial \sqrt{g}}{\partial x^\sigma} \varphi^\sigma = \Delta \sqrt{g}. \\ \\ \left[\sqrt{g} \dfrac{\delta \varphi^\sigma}{\delta x^\sigma} + \dfrac{\delta \sqrt{g}}{\delta x^\sigma} \varphi^\sigma \right] dv_4 = \Delta(\sqrt{g} dv_4) \end{array} \right\}$$

et par conséquent $= 0$, puisque $\sqrt{g}\, dv_4$ est invariant.

En multipliant (4) par dv_4 et en intégrant sur la multiplicité à quatre dimensions on a

$$(6) \qquad \int \left[\frac{\partial R_{\mu\nu}}{\partial x^\sigma} g^{\mu\nu} \sqrt{|g|} \varphi^\sigma + R_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x^\sigma} \sqrt{|g|} \varphi^\sigma \right] dv_4 = \int \frac{\partial}{\partial x^\sigma} [R_{\mu\nu} g^{\mu\nu} \varphi^\sigma \sqrt{|g|}] dv_4,$$

$=$ une intégrale de surface $= 0$, d'après l'hypothèse de la variation.

On tire donc immédiatement de (3) et (6)

$$\int \left[R_{\mu\nu} (g^{\mu\sigma} \varphi^\nu_\sigma + g^{\sigma\nu} \varphi^\mu_\sigma) + \frac{\partial R^{\mu\nu}}{\partial x^\sigma} \varphi^\sigma \right] \sqrt{g} dv_4 = 0$$

Une intégration partielle pour éliminer les dérivées de φ^σ donne immédiatement les identités

$$g^{\mu\nu} \sqrt{g} \frac{\partial R_{\mu\nu}}{\partial x^\sigma} - \frac{d}{dx^\nu} [g^{\mu\nu} R_{\mu\sigma} + g^{\nu\mu} R_{\sigma\mu}] \sqrt{g} = 0$$

identiques au résultat d'Einstein (3). Toutefois, d'une certaine façon, notre résultat est légèrement plus général. On peut dire que le raisonnement d'Einstein est une méthode algébrique, où les connexions entre les coefficients affines et les $g^{\mu\nu}$ sont admises *a priori*, ainsi que les conditions d'intégrabilité et l'hypothèse particulière $\Gamma^\alpha_{\alpha\mu} - \Gamma^\alpha_{\mu\alpha} = 0$.

Notre démonstration est indépendante de ces hypothèses restrictives. En fait, en essayant de déduire les équations générales $R_{\mu\nu} = 0$ et les équations de connexion du principe de variation $\delta \int g^{\mu\nu} R_{\mu\nu} \sqrt{g}\, dv_4 = 0$, on est conduit à la connexion d'Einstein seulement si $\Gamma^\alpha_{\sigma\alpha} - \Gamma^\alpha_{\alpha\sigma} = 0$.

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(3) *Canad. J. Math.*, **2**, 1950, p. 125, équation (176).

D'autre part, en posant *a priori* la connexion d'Einstein, la compatibilité de la variation avec l'hypothèse initiale conduit à $\Gamma_\mu = \Gamma^\alpha_{\mu\alpha} - \Gamma^\alpha_{\alpha\mu} = 0$.

Puisque la démonstration donnée ici n'utilise pas les équations de connexion. elle montre donc nettement que les identités ci-dessus sont indépendantes des équations de connexion. En fait, les $\mathrm{R}_{\mu\nu}$ sont fonctions des Γ seules, et la démonstration des identités de divergence suppose seulement l'existence des dérivées premières des $g^{\mu\nu}$. Elle subsistent même quand par suite des singularités du champ des $g^{\mu\nu}$, il n'existe pas de dérivées secondes ou d'équation de connexion intégrable.

Reprintêd from *Comptes rendus de l'Academie des Sciences* 236, pp. 1333-1335, 1953 (Gauthier – Villars Publishers, Paris).

The Identities of Divergence in the New Unitary Theory

Note of Mr. ***S. N. Bose,*** *presented by Mr. Louis de Broglie*

Einstein(¹) deduced the four identities of divergence from his new theory of Bianchi identities, clearing the ambiguity of covariant differentiation.

The method of Hilbert(²), when slightly modified, also immediately gives the result. Perhaps it is interesting to indicate the mode of calculation.

The relation between the coefficients of affine connection $\Gamma^{\alpha}_{\mu\nu}$ and $g^{\mu\nu}$ is not simple in the new theory. Thus the method of Hilbert regarding the variation of an invariant function of $g^{\mu\nu}$ and its derivatives cannot be directly applied here. In the new theory, this relationship between Γ and g is initially not assumed and the variation gives rise to the equations as well as the equations of connection of Γ and g. The proposed new demonstration takes this into account. The method will be understood according to the following indications.

Let us form the tensor $R_{\mu\nu}$, contracted from the generalized Riemann tensor. Its covariant character is evident from the law of transformation of $\Gamma^{\alpha}_{\mu\nu}$ as in the symmetric theory. $g^{\mu\nu}R_{\mu\nu}$ is thus an invariant under changes of the coordinates, as in the expression $|g|^{1/2}dx_1\,dx_2\,dx_3\,dx_4 = |g|^{1/2}dv_4$.

If $R_{\mu\nu} = 0$ the integral

$$\text{(1)} \qquad \mathrm{I} = \int R_{\mu\nu}\,\delta g^{\mu\nu}\sqrt{g}\,dv_4 = 0$$

for any variation of $g^{\mu\nu}$ alone ; $R_{\mu\nu}$ being a function of the coefficients of affine connection is not changed in the variation. Let us observe with Hilbert (²) that

$$\text{(2)} \qquad p^{\mu\nu} = \frac{\delta g^{\mu\nu}}{\delta x^{\sigma}}\varphi^{\sigma} - g^{\mu\sigma}\varphi^{\nu}_{\sigma} - g^{\sigma\nu}\varphi^{\mu}_{\sigma}$$

obtained from an arbitrary contravariant vector φ^{σ} can be taken for $\delta g^{\mu\nu}$ in (1). Hence

$$\text{(3)} \qquad 0 = \int R_{\mu\nu}\sqrt{|g|}\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\varphi^{\sigma} - g^{\mu\sigma}\varphi^{\nu}_{\sigma} - g^{\sigma\nu}\varphi^{x}_{\sigma}\right)dv_4.$$

The integration is extended over a manifold of four dimensions limited by a manifold of three dimensions and one has everywhere $R_{\mu\nu} = 0$.

We observe that

(¹) *Canad. J. Math.*, **2**, 1950, p. 120 et seq.

(²) *Math. Ann.* **92**, 1924, p. 1 et seq.

$$\frac{d}{dx^{\sigma}}[R_{\mu\nu}\,g^{\mu\nu}\,\varphi^{\sigma}\sqrt{|g|}\,] = \frac{\partial R_{\mu\nu}}{\partial x^{\sigma}}\,g^{\mu\nu}\sqrt{|g|}\,\varphi^{\sigma}+ \tag{4}$$

$$R_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\sqrt{|g|}\,\varphi^{\sigma}+R_{\mu\nu}\,g^{\mu\nu}\left[\sqrt{g}\frac{\partial\varphi^{\sigma}}{\partial x^{\sigma}}+\frac{\partial\sqrt{g}}{\partial x^{\sigma}}\varphi^{\sigma}\right]$$

If we consider $\varphi^{\sigma} \equiv \Delta\, x^{\sigma}$as a suitable small and arbitrary variation of the coordinates which vanishes on the boundary, we observe that

$$\left.\begin{aligned} &\frac{\partial\phi^{\sigma}}{\partial x^{\sigma}}\;dv_4 = \Delta dv_4 \text{ and } \frac{\partial\sqrt{g}}{\partial x^{\sigma}}\,\varphi^{\sigma} = \Delta\sqrt{g}.\\ &\text{and}\\ &\left[\sqrt{g}\;\frac{\delta\varphi^{\sigma}}{\delta x^{\sigma}}+\frac{\delta\sqrt{g}}{\delta x^{\sigma}}\;\varphi^{\sigma}\right]dv_4 = \Delta(\sqrt{g}dv_4)\end{aligned}\right\} \tag{5}$$

and consequently = 0, since $\sqrt{g}\,d\upsilon_4$ is invariant. By multiplying (4) with $d\upsilon_4$ and integrating on the manifold with four dimensions we have

$$\int\left[\frac{\partial R_{\mu\nu}}{\partial x^{\sigma}}\,g^{\mu\nu}\sqrt{|g|}\,\varphi^{\sigma}+R_{\mu\nu}\,\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\sqrt{|g|}\,\varphi^{\sigma}\right]dv_4 = \int\frac{\partial}{\partial x^{\sigma}}\,[R_{\mu\nu}\,g^{\mu\nu}\,\varphi^{\sigma}\sqrt{|g|}\,]dv_4, \tag{6}$$

= a surface integral = 0, as per the hypothesis of variation.
We immediately obtain from (3) and (6)

$$\int\left[R_{\mu\nu}\,(g^{\mu\sigma}\,\varphi^{\nu}_{\sigma}+g^{\sigma\nu}\,\varphi^{\mu}_{\sigma})+\frac{\partial R^{\mu\nu}}{\partial x^{\sigma}}\,\varphi^{\sigma}\right]\sqrt{g}dv_4 = 0$$

A partial integration for eliminating the derivatives of φ^{σ} immediately gives the identities

$$g^{\mu\nu}\sqrt{g}\frac{\partial R_{\mu\nu}}{\partial x^{\sigma}}-\frac{d}{dx^{\nu}}\,[g^{\mu\nu}\,R_{\mu\sigma}+g^{\nu\mu}\,R_{\sigma\mu}]\sqrt{g} = 0$$

identical to the result of Einstein (3). Nevertheless, in a certain way our result is slightly more general. We can say that the reasoning of Einstein is an algebraic method where the connections between the affine coefficients and the $g^{\mu\nu}$ as well as the conditions of integrability and a particular hypothesis $\Gamma^{\alpha}_{\alpha\mu} - \Gamma^{\alpha}_{\mu\alpha} = 0$ are assumed *a priori*.

Our demonstration is independent of these restrictive hypotheses. In fact, while trying to deduce the general equations $R_{\mu\nu} = 0$ and the equations of connection from the principle of variation $\delta\int g^{\mu\nu}\,R_{\mu\nu}\,\sqrt{g}\;d\upsilon_4 = 0$, we are led to the connection of Einstein only if $\Gamma^{\alpha}_{\sigma\alpha} - \Gamma^{\alpha}_{\alpha\sigma} = 0$.

On the other hand, by assuming *a priori* the connection of Einstein, the compatibility of the variation with the initial hypothesis leads to $\Gamma_{\mu} = \Gamma^{\alpha}_{\mu\alpha} - \Gamma^{\alpha}_{\alpha\mu} = 0$.

Since the demonstration given here does not use the equations of connection, it

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(3) *Canad. J. Math.*, 2, 1950,p. 125, equation (176).

clearly shows that the above identities are independent of the equations of connection. In fact, the $R_{\mu\nu}$ are functions of Γ alone, and the demonstration of the identities of divergence only assumes the existence of the first derivatives of $g^{\mu\nu}$. It survives even when due to singularities of the field of $g^{\mu\nu}$, there are no second derivatives or integrable equations of connection.

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UNE THÉORIE DU CHAMP UNITAIRE AVEC $\Gamma_\mu \neq 0$

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Sommaire. — On établit des équations du champ unitaire en faisant varier une intégrale qui satisfait au postulat d'hermiticité. On obtient l'équivalence formelle avec les théories anciennes, soit avec $\Gamma_\mu = 0$, soit avec $\| a_{\lambda\mu} \| = 0$ quand $\Phi_\lambda = 0$; on peut interpréter Φ_λ comme la force de Lorentz si on prend les $a_{\lambda\mu}$ comme les composantes du champ électromagnétique.

Les équations approchées avec composantes antisymétriques prennent la forme maxwellienne pour une certaine valeur des constantes. Toutefois, pour une autre valeur, il est possible d'avoir le champ sans aucune singularité.

1. INTRODUCTION

On peut classer en trois groupes les équations caractéristiques de la théorie du champ d'Einstein :

I. Les équations dué champ qui intéressent uniquement les coefficients de connexion $\Gamma^\lambda_{\mu\nu}$ et leurs derivees.

II. Les équations de connexion qui relient le tenseur g aux coefficients de connexion affine.

III. Les quatre relations de caractére restrictif :

$$\Gamma^\lambda_{\mu\lambda} - \Gamma^\lambda_{\lambda\mu} = 2\Gamma_\mu = 0.$$

Notons que les relations (III) sont semblables aux 24 équations

$$\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0.$$

qui, dans la théorie relativiste de la gravitation sont les conséquences du postulat de la symmétrie de la connexion affine.

Dans la théorie du champ unitaire aussi bien que dans la théorie de la gravitation, les équations due champ (I) et les équations de connexion (II) peuvent se déduire simultanément d'un principe de variation où l'on égale à zéro la variation arbitraire d'une intégrale, la variation étant cepedant assujettie aux restrictions du type (III).

Néanmoins les conditions arbitraries sont bien moins nombreuses dans la théorie unitaire que dans la théorie de la gravitation.

Dans cet article, on essaye d'éliminer toutes les conditions restrictives et d'établir un système d'équations fondamentales en s'appuyant uniquement sur le principe de variation.

On a toutefois utilisé le postulat d'hermiticité récemment énoncé par Einstein pour construire l'intégrale dont la variation conduit à cette nouvelle généralisation.

On verra dans les résultats précisés ci-dessous que des termes additionnels contenant Γ_μ interviennent aussi bien dans les equations du champ que dans les equations de connexion.

Les équations de connexion qui concernent le tenseur g donnent lieu à quelques remarques. Elles preneent ici la forme

$$g^{\mu\nu}_{\lambda}+g^{\mu\alpha}\,\Gamma^{\nu}_{\lambda\alpha}+g^{\alpha\nu}\,\Gamma^{\mu}_{\alpha\lambda} = 3g^{\mu\nu}\,\Phi_{\lambda},$$

tandis que dans les deux théories précédentes le premier membre de cette égalité est nul.

Φ_λ est defini ici par les équations suivantes :

$$\Phi_{\lambda} = \frac{a_{\lambda\beta}\,k^{\beta}}{\sqrt{|\,g\,|}},\ \text{avec}\ a_{\lambda\beta} = \frac{1}{2}\,(g_{\lambda\beta}-g_{\beta\lambda})$$

et

$$k^{\beta} = \frac{1}{3}\,\theta(g^{\beta\nu}+g^{\nu\beta})\Gamma_{\nu}\,.$$

Si $\Gamma_\nu = 0$, $k^\beta = 0$ et par conséquent on a aussi $\Phi = 0$ et l'équation reprend la forme ancienne.

Néanmoins il existe une autre possibilité : que $a_{\alpha\beta}$ soit la partie antisymétrique du tenseur covariant considérée comme liée au vecteur à six composantes (***E***, ***H***) de la théorie électromagnétique.

Si le déterminant

$$\| a_{\lambda\beta} \| = (a_{12}a_{34}+a_{31}a_{24}+a_{23}a_{14})^2 = 0,$$

on peut avoir $\Phi_\lambda = 0$ même si k^μ et Γ_μ ne sont pas nuls.

Remarquons que puisqu'on suppose toujours

$$\| g^{\beta\nu}+g^{\nu\beta} \| \neq 0,$$

k^μ et Γ_μ sont nuls simultanemént.

La condition $\| a\ _\beta \| = 0$ combinée avec la règle habituelle de corrélation citée ci-dessus conduit immédiatement à (***EH***) $= 0$, propriété classique du champ électromagnétique.

On est anisi tenté de poser une corrélation d'une part entre k^μ et le vecteur à quatre dimensions courant-charge (car $k^\mu_\mu = 0$ découle immédiatment des equations générales) et, d'autre part, entre Φ_λ et la force pondéromotrice de Lorentz, en raison de l'identification classique

$$(a_{23}a_{31}\,a_{12})\rightarrow \vec{H}, \qquad (a_{41}\,a_{42}\,a_{43})\rightarrow i\vec{E}.$$

On peut interpréter $\Phi_\lambda = 0$ comme déterminant une distribution stationnaire où la force de Lorentz est nulle, ce qui ramène l'equation de connexion à la forme qu'elle avait dans les théories précédentes.

Dans les équations de champ, l'élement symétrique $P^{\lambda}_{\mu\nu}$ et l'élément antisymétrique $G_{\mu\nu}$ ont des roles essentiellement différents.

Au moyen d'hypothéses sur les ordres de grandeurs relatifs des différents termes, nous avons pu ramener les equations fondamentales à une forme simple.

Physiquement, ces hypothéses impliquent qu'il est possible de négliger la gravitation dans l'étude des relations entre grandeurs électromagnétiques.

Les équations ainsi transformées ont une forme intéressante et l'on peut les interpréter comme de simples conséquences de la théorie de Maxwell si l'on attribue certaines valeurs particulières aux constantes arbitraries. Mais il est possible aussi d'obtenir un champ sans singularités pour d'autres valeurs particuliéres. Le champ n'est alors évidemment pas maxwellien.

Ces aspects encourageants du nouveau système d'équations m'ont amené à penser qu'il convenait de publier les premiers résultats pour provoquer les critiques et les observations.

2. LES NOUVELLES ÉQUATIONS.

Suivant Einstein, toutes les relations de la théoire unitaire doivent satisfaire au postulat d'hermiticité, c'est-à-dire rester invariantes quand $g^{\mu\nu}, g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}$, sontc hangés à la fois en $g^{\nu\mu}$, $g_{\nu\mu}$, $\Gamma^{\lambda}_{\nu\mu}$ par permutation des indices μ et ν.

On voit facilement que dans cette transformation le tenseur d'Einstein $\boldsymbol{E}$ de composantes

$$E_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\xi}_{\mu\nu} \Gamma^{\lambda}_{\xi\lambda} - \Gamma^{\xi}_{\lambda\mu} \Gamma^{\lambda}_{\xi\nu}$$

devient le tenseur $\boldsymbol{H}$ de composantes

$$H_{\nu\mu} = \Gamma^{\lambda}_{\nu\mu,\lambda} - \Gamma^{\lambda}_{\lambda\mu,\nu} + \Gamma^{\epsilon}_{\nu\mu} \Gamma^{\lambda}_{\lambda\epsilon} - \Gamma^{\epsilon}_{\lambda\mu} \Gamma^{\lambda}_{\nu\xi}.$$

Au lieu de l'expression

$$I = g^{\mu\nu} E_{\mu\nu} \sqrt{|g|} = g'^{\mu\nu} E_{\mu\nu}$$

qui intervient dans l'intégrale soumise à variation pour obtenir les équations fondamentales, nous adoptons une forme plus générale

$$I' = \frac{1}{2}(g'^{\mu\nu} E_{\mu\nu} + g'^{\nu\mu} H_{\nu\mu}) + a\gamma^{\mu\nu} \Gamma_{\mu}\Gamma_{\nu} + bA^{\mu\nu} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu})$$

où

$$\gamma^{\mu\nu} = \frac{1}{2}(g'^{\mu\nu} + g'^{\nu\mu}) = \frac{1}{2}\sqrt{|g|}\,(g^{\mu\nu} + g^{\nu\mu})$$

$$A^{\mu\nu} = \frac{1}{2}(g'^{\mu\nu} - g'^{\nu\mu}) = \frac{1}{2}\sqrt{|g|}\,(g^{\mu\nu} - g^{\nu\mu}),$$

et

$$|\bar{g}| = |g|,$$

puispue la permutation de μ et ν ne modifie pas le déterminant. Les termes additionnels vérifient aussi la condition d'hermiticité puisque $\Gamma_{\mu} A^{\mu\nu}$ changent de signe et que $\gamma^{\mu\nu}$ conserve le sien avec la permutation.

Séparons dans les coefficients de connexion affine $\Gamma^{\lambda}_{\mu\nu}$ la partie symétrique $P^{\lambda}_{\mu\nu}$ et la partie antisymétrique $V^{\lambda}_{\mu\nu}$ et posons encore

$$V^{\lambda}_{\mu\nu} = G^{\lambda}_{\mu\nu} - \frac{1}{3}\delta^{\lambda}_{\mu}\Gamma_{\nu} + \frac{1}{3}\delta^{\lambda}_{\nu}\Gamma_{\mu},$$

avec

$$\Gamma_\mu = V^\lambda_{\mu\lambda},$$

on a

$$G^\lambda_{\mu\lambda} = 0.$$

En effectuant les substitutions dans I', tous les termes qui sont produits d'un facteur symétrique par un facteur antisymétrique disparaissent dans la sommation et nous avons finalement

$$I' = \gamma^{\mu\nu} (\bar{E}_{\mu\nu} - G^\lambda_{\mu\varepsilon} G^\varepsilon_{\lambda\nu} + x\Gamma_\mu \Gamma_\nu) + A^{\mu\nu} [G^\lambda_{\mu\nu;\lambda} - y(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu})]$$

où x, y sont des constantes arbitraires,

$$\bar{E}_{\mu\nu} = P^\lambda_{\mu\nu,\lambda} - \frac{1}{2}(P^\lambda_{\mu\lambda,\nu} + P^\lambda_{\nu\lambda,\mu}) + P^\varepsilon_{\mu\nu} P^\lambda_{\varepsilon\lambda} - P^\varepsilon_{\mu\lambda} P^\lambda_{\varepsilon\nu}$$

est la composante symétrisée du tenseur d'Einstein avec connexion affine symétrique et

$$G^\lambda_{\mu\nu;\lambda} = G^\lambda_{\mu\nu,\lambda} - G^\lambda_{\mu\varepsilon} P^\varepsilon_{\lambda\nu} - G^\lambda_{\varepsilon\nu} P^\varepsilon_{\mu\lambda} + G^\varepsilon_{\mu\nu} P^\lambda_{\varepsilon\lambda}$$

est la dérivée covariante classique calculée aussi avec connexion symétrique.

Au cours de la variation arbitraire de

$$L = \int I' dv_4, \quad dv_4 = dx^1 dx^2 dx^3 dx^4.$$

$\gamma^{\mu\nu}$, $A^{\mu\nu}$ conservent automatiquement leur caractére symétrique. Les variations de $\gamma^{\mu\nu}$ et $A^{\mu\nu}$ peuvent donc etre considérées comme arbitraires, ce qui donne immédiatement les équations du champ

$$\frac{\partial I'}{\partial \gamma^{\mu\nu}} = \bar{E}_{\mu\nu} - G^\varepsilon_{\mu\lambda} G^\lambda_{\varepsilon\nu} + x\Gamma_\mu \Gamma_\nu = 0,$$

$$\frac{\partial I'}{\partial A^{\mu\nu}} = G^\lambda_{\mu\nu;\lambda} - y(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}).$$

Pour obtenir les équations de connexion nous égalons d'abord I' à une divergence à quatre dimensions (H) et considérons H seulement puisque la divergence se transforme en une intégrale étendue à un domaine à trois dimensions où tous les coefficients des variations arbitraires s'annulent.

Rappelons-nous cependant que dans la variation des éléments Γ_μ, $P^\lambda_{\mu\nu}$, $G^\lambda_{\mu\nu}$ les quatre relations $G^\lambda_{\mu\lambda} = 0$ restent toujours valables; les 24 composants de $G^\lambda_{\mu\nu}$ ne sont pas arbitraires. Nous devons donc appliquer la méthode habituelle, employer des multiplicateurs indéterminés k^μ et varier la fonction

$$H - 2k^\mu G^\lambda_{\mu\lambda} = H'.$$

Ces coefficients k^μ seront détermines en dernier lieu à partir de l'équation finale.

Nous obtenons ainsi par un calcul facile le systéme d'équations

$$\gamma^{\mu\nu}_{,\lambda} + \gamma^{\mu\alpha} P^\nu_{\lambda\alpha} + \gamma^{\alpha\nu} P^\mu_{\alpha\lambda} - \gamma^{\mu\nu} P^\alpha_{\lambda\alpha} = -[A^{\mu\alpha} G^\nu_{\lambda\alpha} + A^{\alpha\nu} G^\mu_{\alpha\lambda}],$$

$$A^{\mu\nu}_{,\lambda} + A^{\mu\alpha} P^\nu_{\lambda\alpha} + A^{\alpha\nu} P^\mu_{\alpha\lambda} - A^{\mu\nu} P^\alpha_{\lambda\alpha} - k^\mu \delta^\nu_\lambda + k^\nu \delta^\mu_\alpha = -[\gamma^{\mu\alpha} G^\nu_{\lambda\alpha} + \gamma^{\alpha\nu} G^\mu_{\alpha\lambda}],$$

$$yA^{\mu\nu}_{,\nu} + x\gamma^{\mu\nu}\Gamma_\nu = 0,$$

et par conséquent

$$A^{\mu\nu}_{,\nu} = 3k^{\mu},$$

$$\gamma^{\mu\alpha}_{,\alpha} + \gamma^{\alpha\beta} P^{\mu}_{\alpha\beta} + A^{\alpha\beta} G^{\mu}_{\alpha\beta} = 0.$$

D'où on tire immédiatement

$$k^{\mu}_{,\mu} = 0$$

et

$$k^{\mu} = \theta\gamma^{\mu\nu}\Gamma_{\nu}, \quad \text{avec } \theta = -\frac{x}{3y} .$$

Nous pouvons maintenant donner aux équations de connexion la forme convenable : par addition et regroupement on a

$$g'_{\lambda}{}^{\mu\nu} + g'^{\mu\alpha}L^{\nu}_{\lambda\alpha} + g'^{\alpha\nu}L^{\mu}_{\lambda\alpha} - g'^{\mu\nu}L^{\alpha}_{\lambda\alpha} = k^{\mu}\delta^{\nu}_{\lambda} - k^{\nu}\delta^{\mu}_{\lambda},$$

où

$$L^{\nu}_{\lambda\alpha} = G^{\nu}_{\lambda\alpha} + P^{\nu}_{\lambda\alpha},$$

en multipliant par les coefficients $g'_{\mu\nu}$ définis par

$$g'^{\mu\nu}g'_{\mu\lambda} = \delta^{\nu}_{\lambda}, \quad g'^{\mu\nu}g'_{\lambda\nu} = \delta^{\mu}_{\lambda}.$$

nous avons d'abord

$$L'^{\alpha}_{\lambda\alpha} = \frac{1}{2}\ \frac{c,\lambda}{c} + a'_{\lambda\beta}k^{\beta},$$

$$c = |g|$$

et

$$a'_{\lambda\beta} = \frac{1}{2}\ (g'_{\lambda\beta} - g'_{\beta\lambda}),$$

puis en remplacant $L^{\alpha}_{\lambda\alpha}$ par sa valeur et en divisant par $\sqrt{c}$, nous obtenons aprés une transformation facile

$$g^{\mu\nu}_{\lambda} + g^{\mu\alpha}L^{\nu}_{\lambda\alpha} + g^{\alpha\nu}L^{\mu}_{\alpha\lambda} - \frac{g^{\mu\nu}a_{\lambda\beta}k^{\beta}}{\sqrt{c}} = \frac{k^{\mu}\delta^{\nu}_{\lambda} - k^{\nu}\delta^{\mu}_{\lambda}}{\sqrt{c}} .$$

En observant que

$$g^{\mu\alpha}g_{\beta\alpha}k^{\beta} = k^{\mu}, \quad g^{\alpha\nu}g_{\alpha\beta}k^{\beta} = k^{\nu}$$

et en regroupant et additionnant certains termes, nous avons finalement

$$g^{\mu\alpha}_{,\lambda} + g^{\mu\alpha}\,\Gamma'^{\nu}_{\lambda\alpha} + g^{\alpha\nu}\,\Gamma'^{\mu}_{\alpha\lambda} = 3g^{\mu\nu}\frac{a_{\lambda\beta}k^{\beta}}{\sqrt{c}} = 3g^{\mu\nu}\Phi_{\lambda} \ ;$$

où

$$\Gamma'^{\nu}_{\lambda\alpha} = L^{\nu}_{\lambda\alpha} - g_{\beta\alpha}\frac{k\beta}{\sqrt{c}}\,\delta^{\nu}_{\lambda} + g_{\lambda\beta}k^{\beta}\delta^{\nu}_{\alpha},$$

$$\Gamma'^{\mu}_{\alpha\lambda} = L^{\mu}_{\alpha\lambda} - g_{\beta\lambda}\frac{k^{\beta}}{\sqrt{c}}\,\delta^{\mu}_{\alpha} + g_{\alpha\beta}\frac{k^{\beta}}{\sqrt{c}}\,\delta^{\mu}_{\lambda}.$$

Les nouveaux coefficients affines sont

$$\Gamma'^{\lambda}_{\mu\nu} = Q^{\lambda}_{\mu\nu} + U^{\lambda}_{\mu\nu}$$

où

$$Q^{\lambda}_{\mu\nu} = P^{\lambda}_{\mu\nu} + \Phi_{\mu}\delta^{\lambda}_{\nu} + \Phi_{\nu}\delta^{\lambda}_{\mu}$$

et

$$U^{\lambda}_{\mu\nu} = G^{\lambda}_{\mu\nu} - \Psi_{\nu}\delta^{\lambda}_{\mu} + \Psi_{\mu}\delta^{\lambda}_{\nu} \; ;$$

$$\Psi_{\nu} = \frac{s_{\nu\beta}k^{\beta}}{\sqrt{c}} \, , \phi_{\nu} = \alpha_{\nu\beta} \frac{k^{\beta}}{\sqrt{c}} \; ;$$

En introduisant ces nouveaux coefficients affines dans l'équation fondamentale, nous pouvons écrire

$$\bar{E}'_{\mu\nu} + \frac{3}{2}[\Phi_{\mu;\nu} + \Phi_{\nu;\mu}] + 3\Phi_{\mu}\Phi_{\nu} + \frac{x}{\theta^2} b_{\mu a} b_{\nu\beta} \frac{k^{a}k^{\beta}}{c} - G^{\lambda}_{\mu\varepsilon} G^{\varepsilon}_{\lambda\nu} = 0,$$

$$(G^{\lambda}_{\mu\nu})_{;\lambda} - 3G^{\varepsilon}_{\mu\nu}\Phi_{\varepsilon} - \frac{y}{\theta\sqrt{c}}[(b'_{\mu a}k^{a})_{,\nu} - (b'_{\nu a}k^{a})_{,\mu}],$$

avec

$$\tfrac{1}{2} b_{\mu a}(g^{\mu\nu} + g^{\nu\mu}) = \delta^{\nu}_{a}.$$

3. CAS PARTICULAR

Quand $\Phi_{\lambda} = 0$, mais Γ_{μ}, $k^{\mu} \neq 0$, on a

$$a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24} = 0.$$

Effectuons la transformation et observons que

$$\Gamma'_{\mu} = 3\theta\Gamma_{\mu} \; ;$$

$P^{\lambda}_{\mu\nu}$ et $G^{\lambda}_{\mu\nu}$ ne changent pas et nous obtenous

$$\bar{E}'_{\mu\nu} + t\Gamma'_{\mu}\Gamma'_{\nu} - G^{\lambda}_{\mu\varepsilon} G^{\varepsilon}_{\lambda\nu} = 0;$$

$$(G^{\lambda}_{\mu\nu})_{;\lambda} + t(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) = 0,$$

la seule modification est une réduction du nombre des constantes arbitraires.

L'équation de connexion est (puisque $\Phi_{\lambda} = 0$)

$$g^{\mu\lambda}_{\lambda} + g^{\mu a}\Gamma^{\nu}_{\lambda a} + g^{a\nu} \Gamma^{\mu}_{a\lambda} = 0.$$

mais avec $\Gamma_{\mu} \neq 0$.

On montre alors facilement que

$$(A^{\mu\nu}\sqrt{\bar{\Delta}})_{\nu} = \tfrac{1}{2}(g^{\mu a} + g^{a\mu})\Gamma_{a}.$$

4. APPROXIMATION:

Pour trouver une approximation, nous mettons l'équation de connexion sous la forme

$$\gamma^{\mu\nu}_{,\lambda} + \gamma^{\mu a}P^{\nu}_{\lambda a} + \gamma^{a\nu}P^{\mu}_{a\lambda} - \gamma^{\mu\nu}P^{a}_{\lambda a} = -[A^{\mu a}G^{\nu}_{\lambda a} + A^{a\nu} G^{\mu}_{a\lambda}],$$

$$A^{\mu\nu}_{,\lambda} = A^{\mu a}P^{\nu}_{\lambda a} + A^{a\nu}P^{\mu}_{a\lambda} - A^{\mu\nu}P^{a}_{\lambda a} - k^{\mu}\delta^{\nu}_{\lambda} + k^{\nu}\delta^{\mu}_{\lambda} = -[\gamma^{\mu a}G^{\nu}_{\lambda a} + \gamma^{a\nu}G^{\mu}_{a\lambda}].$$

Nous supposons que les coefficients antisymétriques $A^{\mu\nu}$ êt $G^{\lambda}_{\mu\nu}$ sont petits, voisins de zero.

Alors les $P^{\lambda}_{\lambda\mu}$ peuvent aussi étre négligés.

$$\gamma^{\mu\nu} = \delta^{\mu}_{\nu} \, ,$$

la dérivation covariante est remplacée par la derivation ordinaire, et nous avons finalement en premiére approximation

$$A^{\mu\nu}_{,\lambda}-k^{\mu}\delta^{\nu}_{\lambda}+k^{\nu}\delta^{\mu}_{\lambda}=-[G^{\nu}_{\lambda\mu}+G^{\mu}_{\nu\lambda}],$$

$$A^{\mu\nu}_{\nu}=3k^{\mu},$$

d'où l'on déduit facilement

$$(G^{\lambda}_{\mu\nu})=A^{\mu\nu}_{,\lambda}-k^{\mu}\delta^{\nu}_{\lambda}+k^{\nu}\delta^{\mu}_{\lambda}-\tfrac{1}{2}[A^{\mu\nu}_{,\lambda}+A^{\lambda\mu}_{,\nu}+A^{\nu\lambda}_{\mu}],$$

$$(G^{\lambda}_{\mu\nu})_{\lambda}=\frac{2}{3}A^{\mu\nu}_{,\lambda\lambda}-\frac{1}{6}[A^{\mu\nu}_{,\lambda}+A^{\nu\lambda}_{,\mu}+A^{\lambda\mu}_{,\nu}]_{\lambda}$$

et enfin comme

$$\Gamma_{\mu}=\frac{k^{\mu}}{\theta},$$

nous avons le système suivant :

$$A^{\mu\nu}_{\nu}=3k^{\mu}\ ;$$

$$A^{\mu\nu}_{\lambda\lambda}-\frac{1}{4}[A^{\mu\nu}_{\lambda}+A^{\nu\lambda}_{\mu}+A^{\lambda\mu}_{\nu}]_{\lambda}=\frac{3y}{2\theta}[k^{\mu}_{,\nu}-k^{\nu}_{,\mu}].$$

En éliminant k^{μ} nous obtenons

$$A^{\mu\nu}_{\lambda\lambda}\left(1-\frac{y}{2\theta}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{y}{\theta}\right)(A^{\mu\nu}_{\lambda}+A^{\nu\lambda}_{\mu}+A^{\lambda\mu}_{\nu})_{\lambda}=0.$$

Si $1=\dfrac{y}{2\theta}$ nous pouvons poser

$$A^{\mu\nu}_{\lambda}+A^{\nu\lambda}_{\mu}+A^{\lambda\mu}_{\nu}=0$$

et l'équation devient identique aux équations du champ électromagnétique de Maxwell.

Mais si $\dfrac{1}{2}-\dfrac{y}{\theta}=0$, les équations deviennent

$$(A^{\mu\nu}_{\nu})=\rho^{\mu},$$

$$(A^{\mu\nu}_{\lambda}+\ldots)_{\lambda}=-(\rho^{\mu}_{,\nu}-\rho^{\nu}_{,\mu}),$$

avcc

$$A^{\mu\nu}_{\lambda\lambda}=0,$$

système different évidemment de celui de Maxwell, mais qui peut avoir une solntion sans singularitiés en auceun point.

Il est peut être intéressant de remarquer que certaines recherches sur la quantification du champ électromagnétique ont amené de leur côté à poser $A^{\mu\nu}_{\lambda\lambda}=0$, ce qui entrainait implicitement une modification des équations due champ.

Si $\|a_{\lambda\mu}\|=0$ on obtient une approximation présentant les mêmes caractéres.

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A Unitary Field Theory with $\Gamma_\mu \neq 0$

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Summary — The equations of the unitary field are established by varying an integral which satisfies the postulate of hermiticity. One obtains an absolute equivalence with the old theories either with $\Gamma_\mu = 0$ or with $\| a_{\lambda\mu} \| = 0$ when $\Phi_\lambda = 0$; one can interpret Φ_λ as the Lorentz force if $a_{\lambda\mu}$ are taken as the components of the electromagnetic field.

The approximate equations with antisymmetric components take the Maxwellian form for a certain value of the constants. Nevertheless, for another value, it is possible to have a field without any singularity.

1. Introduction

The characteristic equations of Einstein's field theory can be classified into three groups :

I. The field equations which involve only the connection coefficients $\Gamma^\lambda_{\mu\nu}$ and their derivatives.
II. The connection equations which link the tensor *g* to the affine connection coefficients.
III. The four relations of restrictive character

$$\Gamma^\lambda_{\mu\lambda} - \Gamma^\lambda_{\lambda\mu} = 2\Gamma_\mu = 0.$$

Note that the relations in (III) resemble the 24 equations,

$$\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0.$$

which, in the relativistic theory of gravitation, are consequences of the postulate of symmetry of the affine connections.

In unitary field theory as well as in the theory of gravitation the equations of the field (I) and the equations of connection (II) can be simultaneously deduced from the principle of variation where the arbitrary variation of an integral is equal to zero, the variation being subjected to restrictions of type (III).

Nevertheless, the arbitrary conditions are much less numerous in the unitary theory than in the theory of gravitation.

In this article we are trying to eliminate all the restrictive conditions and establish a system of fundamental equations by relying only on the principle of variation.

We have used the postulate of hermiticity recently enunciated by Einstein to construct the integral, the variation of which leads to this new generalization.

The specific results given below will show that the additional terms containing Γ_μ occur as well in the field equations as in the equations of connection.

The equations of connection which concern the tensor *g* give rise to certain remarks.

Here they take the form

$$g^{\mu\nu}_{\lambda}+g^{\mu\alpha}\,\Gamma^{\nu}_{\lambda\alpha}+g^{\alpha\nu}\,\Gamma^{\mu}_{\alpha\lambda} = 3g^{\mu\nu}\,\Phi_{\lambda},$$

whereas in the two proceeding theories the first member of this equality is zero.

Here Φ_λ is defined by the following equation

$$\Phi_{\lambda} = \frac{a_{\lambda\beta}\,k^{\beta}}{\sqrt{|\,g\,|}},\ \text{with}\ q_{\lambda\beta} = \frac{1}{2}\,(g_{\lambda\beta}-g_{\beta\lambda})$$

and

$$k^{\beta} = \frac{1}{3}\,\theta(g^{\beta\nu}+g^{\nu\beta})\Gamma_{\nu}.$$

If $\Gamma_\nu = 0$, $k^\beta = 0$ and consequently if we also have $\Phi = 0$, the equation again takes the old form.

However, there exits another possibility : that $a_{\alpha\beta}$ being the antisymmetric part of the covariant tensor, is considered as linked to a vector with the six components (**E**, **H**) of the electromagnetic theory.

If the determinant

$$\|\,a_{\lambda\beta}\,\| = (a_{12}a_{34}+a_{31}a_{24}+a_{23}a_{14})^{2} = 0,$$

we can have $\Phi_\lambda = 0$ even if k^μ and Γ_μ are not zero.

Note that since we always suppose

$$\|\,g^{\beta\nu}+g^{\nu\beta}\,\| \neq 0,$$

k^μ and Γ_μ are simultaneously zero.

The condition $\|\,a_{\lambda\beta}\,\| = 0$ combined with the usual rule of correlation just mentioned immediately leads to (**E H**) = 0, a classical property of the electromagnetic field.

One is also tempted to put a correlation on the one hand between k^μ and the four dimensional charge current vector (because $k^{\mu}_{,\mu} = 0$ comes immediately from the general equations) and on the other hand, between Φ_λ and the ponderomotive force of Lorentz, due to the classical identifications

$$(a_{23}a_{31}\;a_{12})\rightarrow \vec{H}, \qquad (a_{41}\;a_{42}\;a_{43})\rightarrow i\vec{E}.$$

We can interpret $\Phi_\lambda = 0$ as determining a stationary distribution, where the Lorentz force is zero. This brings back the equation of connection to the form which was found in previous theories.

In the field equations the symmetric element $P^{\lambda}_{\mu\nu}$ and the antisymmetric element $G^{\lambda}_{\mu\nu}$ have totally different roles.

By means of hypotheses on the orders of relative magnitude of the various terms we could bring back the fundamental equations to a simple form.

Physically, these hypotheses imply that it is possible to neglect gravitation in a study of relations between electromagnetic quantities.

The equations thus transformed have an interesting form and we can interpret them as simple consequences of the Maxwell theory if we attribute particular values to the arbitrary constants.

But it is also possible to obtain a field without any singularities for other particular values. The field is then not Maxwellian.

These encouraging aspects from the new system of equations have made me think that it would be appropriate to publish the first findings in order to provoke criticisms and observations.

2. The new equations

Following Einstein, all the relations of unitary field theory must satisfy the postulate of hermiticity, i.e. remain unchanged when $g^{\mu\nu}$, $g_{\mu\nu}$, $\Gamma^\lambda_{\mu\nu}$ are changed to $g^{\nu\mu}$, $g_{\nu\mu}$, $\Gamma^\lambda_{\nu\mu}$ all together through permutations of the indices μ and ν.

We can easily see that in this transformation, the Einstein tensor **E** of components

$$E_{\mu\nu} = \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\xi_{\mu\nu}\,\Gamma^\lambda_{\xi\lambda} - \Gamma^\xi_{\lambda\mu}\,\Gamma^\lambda_{\xi\nu}$$

become the tensor **H** of components

$$H_{\nu\mu} = \Gamma^\lambda_{\nu\mu,\lambda} - \Gamma^\lambda_{\lambda\mu,\nu} + \Gamma^\epsilon_{\nu\mu}\,\Gamma^\lambda_{\lambda\epsilon} - \Gamma^\epsilon_{\lambda\mu}\,\Gamma^\lambda_{\nu\xi}.$$

Instead of the expression

$$I = g^{\mu\nu}\,E_{\mu\nu}\sqrt{|g|} = g'^{\mu\nu}\,E_{\mu\nu}$$

which occurs in the integral which is subjected to variation in order to obtain the fundamental equations, we adopt a form which is more general

$$I' = \frac{1}{2}(g'^{\mu\nu}\,E_{\mu\nu} + g'^{\nu\mu}\,H_{\nu\mu}) + a\gamma^{\mu\nu}\,\Gamma_\mu\Gamma_\nu + bA^{\mu\nu}\,(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu})$$

where

$$\gamma^{\mu\nu} = \frac{1}{2}(g'^{\mu\nu} + g'^{\nu\mu}) = \frac{1}{2}\sqrt{|g|}\ (g^{\mu\nu} + g^{\nu\mu})$$

$$A^{\mu\nu} = \frac{1}{2}(g'^{\mu\nu} - g'^{\nu\mu}) = \frac{1}{2}\sqrt{|\bar{g}|}\,(g^{\mu\nu} - g^{\nu\mu}),$$

and

$$|\bar{g}| = |g|,$$

since a permutation of μ and ν does not modify the determinant. The additional terms also satisfy the condition of hermicity since Γ_μ $A^{\mu\nu}$ changes sign and $\gamma^{\mu\nu}$ retains its sign with permutation. Let us separate the symmetric part $P^\lambda_{\mu\nu}$ and the antisymmetric part $V^\lambda_{\mu\nu}$ in the affine connection coefficients $\Gamma^\lambda_{\mu\nu}$ and let us also put

$$V_{\mu\nu} = G^\lambda_{\mu\nu} - \frac{1}{3}\,\delta^\lambda_\mu\Gamma_\nu + \frac{1}{3}\delta^\lambda_\nu\,\Gamma_\mu,$$

with

$$\Gamma_\mu = V^\lambda_{\mu\lambda},$$

one has

$$G^{\lambda}_{\mu\lambda} = 0.$$

Through substitution in I′, all the terms that are products of the symmetric factor and the antisymmetric factor disappear in the summation, and we finally have

$$I' = \gamma^{\mu\nu}(\bar{E}_{\mu\nu} - G^{\lambda}_{\mu\varepsilon}\, G^{\varepsilon}_{\lambda\nu} + x\Gamma_{\mu}\,\Gamma_{\nu}) + A^{\mu\nu}\,[G^{\lambda}_{\mu\nu;\lambda} - y(\Gamma_{\mu,\nu} - \Gamma_{\nu\,\mu})]$$

where x, y are arbitrary constants.

$$\bar{E}_{\mu\nu} = P^{\lambda}_{\mu\nu,\lambda} - \frac{1}{2}(P^{\lambda}_{\mu\lambda,\nu} + P^{\lambda}_{\nu\lambda,\,\mu}) + P^{\varepsilon}_{\mu\nu}\, P^{\lambda}_{\varepsilon\lambda} - P^{\varepsilon}_{\mu\lambda}\, P^{\lambda}_{\varepsilon\nu}$$

is the symmetrized component of the Einstein tensor with symmetric affine connection and

$$G^{\lambda}_{\mu\nu;\lambda} = G^{\lambda}_{\mu\nu,\lambda} - G^{\lambda}_{\mu\varepsilon}P^{\varepsilon}_{\lambda\nu} - G^{\lambda}_{\varepsilon\nu}P^{\varepsilon}_{\mu\lambda} + G^{\varepsilon}_{\mu\nu}P^{\lambda}_{\varepsilon\lambda}$$

is the classical covariant derivative, also calculated with symmetric connection.

During the course of arbitrary variation of

$$L = \int I' d\nu_4, \quad d\nu_4 = dx^1 dx^2 dx^3 dx^4.$$

$\gamma^{\mu\nu}$, $A^{\mu\nu}$ automatically retain their symmetry character. The variations of $\gamma^{\mu\nu}$ and $A^{\mu\nu}$ can thus be considered as arbitrary, which immediately gives rise to the field equations

$$\frac{\partial I'}{\partial \gamma^{\mu\nu}} = \bar{E}_{\mu\nu} - G^{\varepsilon}_{\mu\lambda}G^{\lambda}_{\varepsilon\nu} + x\Gamma_{\mu}\Gamma_{\nu} = 0,$$

$$\frac{\partial I'}{\partial A^{\mu\nu}} = G^{\lambda}_{\mu\nu;\lambda} - y(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}).$$

In order to obtain the equations of connection, let us first make I′ equal to a divergence in four dimensions (H) and let us consider only H since the divergence transforms itself to an integral over a three dimensional domain where all the coefficients of arbitrary variations cancel each other.

Let us recall the fact that in the variation of the elements Γ_{μ}, $P^{\lambda}_{\mu\nu}$, $G^{\lambda}_{\mu\nu}$ the four relations $G^{\lambda}_{\mu\lambda} = 0$ always remain valid ; the 24 component of $G^{\lambda}_{\mu\nu}$ are not arbitrary. We can thus apply the usual method of employing undermined multipliers k^{μ} and varying the function

$$H - 2k^{\mu}G^{\lambda}_{\mu\lambda} = H'.$$

These coefficients k^{μ} will be determined in the end from the final equation.

Thus through easy calculation we obtain a system of equations

$$\gamma^{\mu\nu}_{,\lambda} + \gamma^{\mu\alpha}P^{\nu}_{\lambda\alpha} + \gamma^{\alpha\nu}P^{\mu}_{\alpha\lambda} - \gamma^{\mu\nu}P^{\alpha}_{\lambda\alpha} = -[A^{\mu\alpha}G^{\nu}_{\lambda\alpha} + A^{\alpha\nu}G^{\mu}_{\alpha\lambda}],$$

$$A^{\mu\nu}_{,\lambda} + A^{\mu\alpha}P^{\nu}_{\lambda\alpha} + A^{\alpha\nu}P^{\mu}_{\alpha\lambda} - A^{\mu\nu}P^{\alpha}_{\lambda\alpha} - k^{\mu}\delta^{\nu}_{\lambda} + k^{\nu}\delta^{\mu}_{\alpha} = -[\gamma^{\mu\alpha}G^{\nu}_{\lambda\alpha} + \gamma^{\alpha\nu}G^{\mu}_{\alpha\lambda}],$$

$$yA^{\mu\nu}_{\nu} + x\gamma^{\mu\nu}\Gamma_{\nu} = 0,$$

and consequently

$$A^{\mu\nu}_{,\nu} = 3k^{\mu},$$

$$\gamma^{\mu a}_{,a} + \gamma^{a\beta} P^{\mu}_{a\beta} + A^{a\beta} G^{\mu}_{a\beta} = 0.$$

From where we immediately obtain

$$k^{\mu}_{,\mu} = 0$$

and

$$k^{\mu} = \theta\gamma^{\mu\nu}\Gamma_{\nu}, \quad \text{avec } \theta = -\frac{x}{3y} .$$

We can now give a suitable form to the equations of connection : through addition and regrouping we have

$$g'_{\lambda}{}^{\mu\nu} + g'^{\mu a}L^{\nu}_{\lambda a} + g'^{a\nu}L^{\mu}_{\lambda a} - g'^{\mu\nu}L^{a}_{\lambda a} = k^{\mu}\delta^{\nu}_{\lambda} - k^{\nu}\delta^{\mu}_{\lambda},$$

where

$$L^{\nu}_{\lambda a} = G^{\nu}_{\lambda a} + P^{\nu}_{\lambda a},$$

by multiplying with the coefficient $g'_{\mu\nu}$ defined by

$$g'^{\mu\nu}g'_{\mu\lambda} = \delta^{\nu}_{\lambda}, \quad g'^{\mu\nu}g'_{\lambda\nu} = \delta^{\mu}_{\lambda}.$$

we first obtain

$$L'^{a}_{\lambda a} = \frac{1}{2} \; \frac{c, \lambda}{c} + a'_{\lambda\beta}k^{\beta},$$

$$c = |g|$$

and

$$a'_{\lambda\beta} = \frac{1}{2} \; (g'_{\lambda\beta} - g'_{\beta\lambda}),$$

then, by replacing $L^{a}_{\lambda a}$ by its value and dividing it by $\sqrt{c}$, we get an easy transformation

$$g^{\mu\nu}_{\lambda} + g^{\mu a}L^{\nu}_{\lambda a} + g^{a\nu}L^{\mu}_{a\lambda} - \frac{g^{\mu\nu}a_{\lambda\beta}k^{\beta}}{\sqrt{c}} = \frac{k^{\mu}\delta^{\nu}_{\lambda} - k^{\nu}\delta^{\mu}_{\lambda}}{\sqrt{c}} .$$

By observing that

$$g^{\mu a}g_{\beta a}k^{\beta} = k^{\mu}, \quad g^{a\nu}g_{a\beta}k^{\beta} = k^{\nu}$$

and by regrouping and adding certain terms, we finally obtain

$$g^{\mu\nu}_{,\lambda} + g^{\mu a}\,\Gamma'^{\nu}_{\lambda a} + g^{a\nu}\,\Gamma'^{\mu}_{a\lambda} = 3g^{\mu\nu}\frac{a_{\lambda\beta}k^{\beta}}{\sqrt{c}} = 3g^{\mu\nu}\Phi_{\lambda} \;;$$

where

$$\Gamma'^{\nu}_{\lambda a} = L^{\nu}_{\lambda a} - g_{\beta a}\,\frac{k\beta}{\sqrt{c}}\,\delta^{\nu}_{\lambda} + g_{\lambda\beta}k^{\beta}\delta^{\nu}_{a},$$

$$\Gamma'^{\mu}_{a\lambda} = L^{\mu}_{a\lambda} - g_{\beta\lambda}\,\frac{k^{\beta}}{\sqrt{c}}\,\delta^{\mu}_{a} + g_{a\beta}\,\frac{k^{\beta}}{\sqrt{c}}\,\delta^{\mu}_{\lambda}.$$

The new affine coefficients are

$$\Gamma'^{\lambda}_{\mu\nu} = Q^{\lambda}_{\mu\nu} + U_{\mu\nu}$$

where

$$Q^{\lambda}_{\mu\nu} = P^{\lambda}_{\mu\nu} + \Phi_{\mu}\delta^{\lambda}_{\nu} + \Phi_{\nu}\delta^{\lambda}_{\mu}$$

and

$$U^{\lambda}_{\mu\nu} = G^{\lambda}_{\mu\nu} - \Psi_{\nu}\delta^{\lambda}_{\mu} + \Psi_{\mu}\delta^{\lambda}_{\nu} \; ;$$

$$\Psi_{\nu} = \frac{s_{\nu\beta}k^{\beta}}{\sqrt{c}} \, , \phi_{\nu} = a_{\nu\beta}\frac{k^{\beta}}{\sqrt{c}} \; ;$$

By introducing these new affine coefficients in the fundamental equation, we can write

$$\bar{E}'_{\mu\nu} + \frac{3}{2}[\Phi_{\mu;\nu} + \Phi_{\nu;\mu}] + 3\Phi_{\mu}\Phi_{\nu} + \frac{x}{\theta^{2}}\, b_{\mu\alpha}\, b_{\nu\beta}\frac{k^{\alpha}k^{\beta}}{c} - G^{\lambda}_{\mu\varepsilon}\, G^{\varepsilon}_{\lambda\nu} = 0,$$

$$(G^{\lambda}_{\mu\nu})_{;\lambda} - 3G^{\varepsilon}_{\mu\nu}\Phi_{\varepsilon} - \frac{y}{\theta\sqrt{c}}[(b'_{\mu\alpha}k^{\alpha})_{,\nu} - (b'_{\nu\alpha}k^{\alpha})_{,\mu}],$$

with

$$\tfrac{1}{2}\, b_{\mu\alpha}(g^{\mu\nu} + g^{\nu\mu}) = \delta^{\nu}_{\alpha}.$$

3. Particular case

When $\Phi_{\lambda} = 0$ but Γ_{μ} , $k^{\mu} \neq 0$ we have

$$a_{12}a_{34} + a_{23}a_{14} + a_{31}a_{24} = 0.$$

Let us make the transformation and observe that

$$\Gamma'_{\mu} = 3\theta\Gamma_{\mu} \; ;$$

$P^{\lambda}_{\mu\nu}$ and $G^{\lambda}_{\mu\nu}$ do not change and we obtain

$$E'_{\mu\nu} + t\Gamma'_{\mu}\Gamma'_{\nu} - G^{\lambda}_{\mu\varepsilon}\, G^{\varepsilon}_{\lambda\nu} = 0;$$

$$(G^{\lambda}_{\mu\nu})_{;\lambda} + t(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) = 0,$$

the only modification is a reduction of the number of arbitrary constants.

The equation of connection is (since $\Phi_{\lambda} = 0$)

$$g^{\mu\lambda}_{\lambda} + g^{\mu\alpha}\Gamma^{\nu}_{\lambda\alpha} + g^{\alpha\nu}\, \Gamma^{\mu}_{\alpha\lambda} = 0.$$

but with $\Gamma_{\mu} \neq 0$.

Thus it is easily shown that

$$(A^{\mu\nu}\sqrt{\Delta})_{\nu} = \tfrac{1}{2}\,(g^{\mu\alpha} + g^{\alpha\mu})\Gamma_{\alpha}.$$

4. Approximation

In order to find an approximation we put the equation of connection in the form

$$\gamma^{\mu\nu}_{,\lambda} + \gamma^{\mu a}P^{\nu}_{\lambda a} + \gamma^{a\nu}P^{\mu}_{a\lambda} - \gamma^{\mu\nu}P^{a}_{\lambda a} = -[A^{\mu a}G^{\nu}_{\lambda a} + A^{a\nu}\,G^{\mu}_{a\lambda}],$$
$$A^{\mu\nu}_{,\lambda} = A^{\mu a}P^{\nu}_{\lambda a} + A^{a\nu}P^{\mu}_{a\lambda} - A^{\mu\nu}P^{a}_{\lambda a} - k^{\mu}\delta^{\nu}_{\lambda} + k^{\nu}\delta^{\mu}_{\lambda} = -[\gamma^{\mu a}G^{\nu}_{\lambda a} + \gamma^{a\nu}G^{\mu}_{a\lambda}].$$

Let us suppose that the antisymmetric coefficients $A^{\mu\nu}$ and $G^{\lambda}_{\mu\nu}$ are small, close to zero.

Thus $P^{\nu}_{\lambda\mu}$ can also be neglected :

$$\gamma^{\mu\nu} = \delta^{\mu}_{\nu},$$

the covariant derivative is replaced by an ordinary derivative, and we finally obtain in the first approximation

$$A^{\mu\nu}_{,\lambda} - k^{\mu}\delta^{\nu}_{\lambda} + k^{\nu}\delta^{\mu}_{\lambda} = -[G^{\nu}_{\lambda\mu} + G^{\mu}_{\nu\lambda}],$$
$$A^{\mu\nu}_{\nu} = 3k^{\mu},$$

from which we can easily deduce

$$(G^{\lambda}_{\mu\nu}) = A^{\mu\nu}_{,\lambda} - k^{\mu}\delta^{\nu}_{\lambda} + k^{\nu}\delta^{\mu}_{\lambda} - \tfrac{1}{2}\,[A^{\mu\nu}_{,\lambda} + A^{\lambda\mu}_{,\nu} + A^{\nu\lambda}_{\mu}],$$

$$(G^{\lambda}_{\mu\nu})_{\lambda} = \frac{2}{3}\,A^{\mu\nu}_{,\lambda\lambda} - \frac{1}{6}[A^{\mu\nu}_{,\lambda} + A^{\nu\lambda}_{,\mu} + A^{\lambda\mu}_{,\nu}]_{\lambda}$$

and finally with

$$\Gamma_{\mu} = \frac{k^{\mu}}{\theta},$$

we obtain the following system

$$A^{\mu\nu}_{\nu} = 3k^{\mu}\ ;$$

$$A^{\mu\nu}_{\lambda\lambda} - \frac{1}{4}[A^{\mu\nu}_{\lambda} + A^{\nu\lambda}_{\mu} + A^{\lambda\mu}_{\nu}]_{\lambda} = \frac{3y}{2\theta}\,[k^{\mu}_{,\nu} - k^{\nu}_{,\mu}].$$

By eliminating k^{μ} we obtain

$$A^{\mu\nu}_{\lambda\lambda}\left(1 - \frac{y}{2\theta}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{y}{\theta}\right)(A^{\mu\nu}_{\lambda} + A^{\nu\lambda}_{\mu} + A^{\lambda\mu}_{\nu})_{\lambda} = 0.$$

If $1 = \frac{y}{2\,\theta}$ we can put

$$A^{\mu\nu}_{\lambda} + A^{\nu\lambda}_{\mu} + A^{\lambda\mu}_{\nu} = 0$$

and the equation becomes identical to Maxwell's equations of electromagnetic fields.

But if $\frac{1}{2} - \frac{y}{\theta} = 0$, the equations become

$$(A^{\mu\nu}_{\nu}) = \rho^{\mu},$$

with

$$(A^{\mu\nu}_{\lambda} + \ldots)_{\lambda} = -(\rho^{\mu}_{,\nu} - \rho^{\nu}_{,\mu}),$$
$$A^{\mu\nu}_{\lambda\lambda} = 0,$$

the system being evidently different from that of Maxwell, but which can have a solution without singularities at any point.

It is perhaps interesting to note that certain researches on the quantization of the electromagnetic field have put the subsidiary condition $A^{\mu}_{\lambda}{}^{\nu}_{\lambda} = 0$, which would imply a modification of field equations.

If $\| a_{\lambda\mu} \| = 0$, we obtain an approximation presenting the same characteristics.

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CERTAINES CONSÉQUENCES DE L'EXISTENCE DU TENSEUR g DANS LE CHAMP AFFINE RELATIVISTE

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Sommaire. — On établit des relations où entrent seulement $\Gamma^{\lambda}_{\mu\nu}$ et leurs dérivées à partir des conditions d'intégrabilité des équations auxquelles satisfait le tenseur g des théories relativistes.

Le nombre de relations indépendantes entre éléments du champ est bien plus grand en théorie unitaire qu'en théorie de la gravitation.

1. INTRODUCTION

Le formalisme abstrait de la théorie de la relativité a actuellement abouti à la conception d'un champ de coefficients de connexion affine d'une variété 4-dimensionnelle.

Soient δx^{μ} les composantes d'un déplacement parallele et infiniment petit auquel un vecteur A de la variété considérée est assujetti: les variations des composantes A^{μ} qui en résultent sont données par la règle

$$\delta A^{\mu} = - A^{\alpha}\Gamma^{\mu}_{\alpha\beta}\ \delta x^{\beta} \qquad \ldots \quad (1.1)$$

Dans la théorie de la gravitation les termes $\Gamma^{\mu}_{\alpha\beta}$ sont symétriques, ou $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$; donc, la loi de transport y est unique.

Au contraire, comme les coefficients sont dissymétriques dans la théorie unitaire, on peut y avoir aussi une deuxiéme loi de transport, savoir :

$$\bar{\delta} A^{\mu} = -\delta x^{\alpha}\ \Gamma^{\mu}_{\alpha\beta} A^{\beta}. \qquad \ldots \quad (1.2)$$

En s'appuyant sur la rele (1.1), on peut calculer facilement le tenseur de courbure $\boldsymbol{R}$ dont les composantes

$$\left.\begin{array}{l} \underset{\nu}{R^{\lambda}_{\alpha\mu}} = \Gamma^{\lambda}_{\alpha\mu,\,\nu} - \Gamma^{\lambda}_{\alpha\nu,\,\mu} + \Gamma^{\varepsilon}_{\alpha\mu}\Gamma^{\lambda}_{\varepsilon\nu} - \Gamma^{\varepsilon}_{\alpha\nu}\Gamma^{\lambda}_{\varepsilon\mu} \\ \\ \left(\text{ou } \Gamma^{\lambda}_{\alpha u,\nu} = \dfrac{d}{dx^{\nu}}\Gamma^{\lambda}_{\alpha\mu}\,, \ldots \right) \end{array}\right\} \qquad (1.3)$$

sont antisymétriques en indices μ et v : et ensuite, par voie de contraction, on a facilement le tenseur d'Einstein $\boldsymbol{E}$, avec les composantes

$$E_{\alpha\mu} = \Gamma^{\lambda}_{\alpha\mu,\,\lambda} - \Gamma^{\lambda}_{\alpha\lambda,\,\mu} + \Gamma^{\varepsilon}_{\alpha\mu}\Gamma^{\lambda}_{\varepsilon\lambda} - \Gamma^{\varepsilon}_{\alpha\lambda}\Gamma^{\lambda}_{\varepsilon\mu} \qquad (1.4)$$

Mais la théorie unitaire adment aussi (1.2) comme loi du transport; donc, le meme raisonnement que ci-dessus nous donnera, à partire de (1.2) un autre transport S (substitué à $\boldsymbol{R}$) avec des composantes

$$\underset{\nu}{S^{\lambda}_{\mu\alpha}} = \Gamma^{\lambda}_{\mu\alpha,\,\nu} - \Gamma^{\lambda}_{\nu\alpha,\,\mu} + \Gamma^{\varepsilon}_{\mu\alpha}\Gamma^{\lambda}_{\nu\varepsilon} - \Gamma^{\varepsilon}_{\nu\alpha}\Gamma^{\lambda}_{\mu\varepsilon} \qquad (1.5)$$

et finalement aussi le tenseur ***H*** (substitute à ***E*** Einstein) avec des composantes

$$H_{\mu\alpha} = \Gamma^{\lambda}_{\mu\alpha,\lambda} - \Gamma^{\lambda}_{\lambda\alpha,\mu} \mp \Gamma^{\varepsilon}_{\mu\alpha}\Gamma^{\lambda}_{\lambda\varepsilon} - \Gamma^{\varepsilon}_{\lambda\alpha}\Gamma^{\lambda}_{\mu\varepsilon}. \tag{1.6}$$

Les équations $E_{\alpha\mu} = 0$ sont celles du champ dans la théori de la gravitation.

L'ambiguité de la loi du transport dans la théorie unitaire opposeapparement des difficulties au choix de $E_{\alpha\mu} = 0$ comme équations fonndamentales.

Mais celles-ci ont été tournées par Einstein de la manière suivante : il impose au champ affine quatre conditions restrictives savoir

$$\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\mu} = 2\Gamma_{\mu} = 0. \tag{1.7}$$

Bein qu'arbitraire, cette hypothèse donne facilement

$$\boldsymbol{E} = \boldsymbol{H}\,;$$

les deux règles conduisent ainsi au même tenseur; donc les equations du champ peuvent rester les mêmes dans lés deux théories.

A côté du systeme

$$E_{\mu\nu} = 0 \tag{A}$$

qui dépend seulement des coefficients $\Gamma^{\lambda}_{\mu\nu}$ et de leurs dérivées premieres, il y a aussi un second système dans lequel entrent également les composantes du potentiel $g^{\mu\nu}$ et leurs dérivées : ce sont en effet les équations

$$g^{\mu\nu}_{,k} + g^{\mu\lambda}\Gamma^{\nu}_{k\lambda} + g^{\lambda\nu}\Gamma^{\mu}_{\lambda k} = 0. \tag{B}$$

On a démontré que (A) et (B) se laissent déduire d'un principe variationnel.

On forme l'integrale invariante

$$I = \int \sqrt{|g|}\, g^{\mu\nu}E_{\mu\nu}dV_4$$

avec des composantes du potentiel et du tenseur d'Einstein (supposé unique à cause de $\Gamma_{\mu} = 0$) et l'on pose

$$\delta I = 0$$

pour toutes variations arbitraries de $g^{\mu\nu}$ et $\Gamma^{\lambda}_{\mu\nu}$; on obtient immediatement les équations cherchées.

II est évient que l'existence du tenseur de potentiel dans le champ affine donnerait à celui-ci un caractere tout à fait particulier.

En effet (B), regardée comme un système d'équations différentielles, permettra de déterminer $g^{\mu\nu}$ seulement quand les conditions d'intégrabilité seront satisfaites.

Ainsi, les relations qui suivent comme conséquences de cette hypothéses d'intégrabilité entre les coefficients $\Gamma^{\lambda}_{\mu\nu}$ et leurs dérivées exprimerontcelles des propriétés caractéristiques du champ, qui sont postulées implicitement par la théorie dè la relativité comme par la théorie actuelle du champ unitaire.

Les obtenir sous une forme explicite est précisément le but de ce travail.

2. Cas I. Quand les coefficients $\Gamma^{\lambda}_{\mu\nu}$ sont symétriques, les conditions d'intégrailité des équations (B) savoir :

$$g^{\mu\nu}_{,k} + g^{\mu\lambda}\Gamma^{\nu}_{k\lambda} + g^{\lambda\nu}\Gamma^{\mu}_{\lambda k} = 0$$

sont plus facilement exprimables par l'intermédiaire du tenseur de courbure $\boldsymbol{R}$.

Si les composantes $R_{\underset{n}{lm}}$ sont exprimées par les multiples $g^{\mu\nu}$ et des parenthèses à quatre indices comme

$$R^{\lambda}_{\underset{n}{lm}} = g^{\lambda k}\,(klmn), \qquad \ldots \quad (2.1)$$

les conditions d'intégrabilité seront bien exprimées par les propriétés suivantes et bien connues de ces parenthèses, savoir :

$$(klmn) = -(lkmn) = (mnkl) = -(nmkl). \qquad \ldots \quad (2.2)$$

A partir de celles-ci, on peut èliminer simultanément $g^{\mu\nu}$ et les parenthèses et obtenir des relations entre les composantes de R seulement.

(i) Comme

$$R^{\lambda}_{\underset{n}{lm}} = g^{\lambda k}\,(klmn),$$

on pose $\lambda = l$ et l'on contracte. On observe que les parenthèses sont antisymétriques en indices (k et l) à côté de g^{lk} qui sont symétriques.

On a alors automatiquement :

$$R^{l}_{\underset{n}{lm}} = 0 \qquad \ldots \quad (2.3)$$

(ii) De même, comme

$$R^{\lambda}_{\underset{n}{lm}}R^{n}_{\underset{\lambda}{l'm'}} = g^{\lambda k}\,(klmn)\,g^{nk'}\,(k'l'm'\lambda),$$

et

$$R^{k'}_{\underset{k}{ml}}R^{k}_{\underset{k'}{m'l'}} = g^{k'n}\,(nmlk)\,g^{k\lambda}\,(\lambda\, m'l'k'),$$

nous avons

$$R^{\lambda}_{\underset{n}{lm}}R^{n}_{\underset{\lambda}{l'm'}} = R^{k'}_{\underset{k}{ml}}\,R^{k}_{\underset{k'}{m'l'}} \qquad \ldots \quad (2.4)$$

à cause des relations (2.2) (en ce qui concerne les parentheses).

(iii) On peut ecrire les 96 composantes du tenseur $R^{\lambda}_{\underset{m}{kl}}$ explicitement en fonction de Γ^{λ}_{mn} et de leurs dérivées, à savoir (1.3), d'où découlent aisément les relations

$$R^{\lambda}_{\underset{n}{lm}} + R^{\lambda}_{\underset{l}{mn}} + R^{\lambda}_{\underset{m}{nl}} = 0. \qquad \ldots \quad (2.5)$$

(2.3), (2.4), (2.5) sont des propriétés caractéristiques du champ symétrique, exprimées par les composantes du tenseur de courbure ***R***.

3. Cas II. Quand le champ est dissymétrique, les conditions d'intégrabilité des équations (B) peuvent aussi se déduire avec facilité.

Avec (B) on considère aussi le système de leurs dérivées premières (B_1); entre (B) et (B_1) on élimine toutes les dérivées premiéres de $g^{\mu\nu}$ et en ajoutant les conditions

$$g^{\mu\nu}_{,l,k} - g^{\mu\nu}_{,k,l} = 0,$$

on arrive immédiatement aux conditions d'intégrabilité suivantes :

$$0 = g^{a\lambda} \underset{n}{R^{\mu}_{am}} + g^{\mu d} \underset{n}{S^{\lambda}_{md}}, \qquad \ldots \quad (3.1)$$

d'où l'on peut déduire immédiatement celles-ci :

(i) En multipliant (3.1) par $g_{\mu\lambda}$ et en sommant ensuite (comme $g^{a\lambda} g_{\mu\lambda} = \delta^{a}_{\mu}$, $g^{\mu d} g_{\mu\lambda} = \delta^{d}_{\lambda}$ et δ' sont les symboles de Kronecker), on a

$$\underset{n}{R^{a}_{am}} + \underset{n}{S^{d}_{md}} = 0$$

$$\frac{d}{dx^{m}} (\Gamma^{a}_{na} + \Gamma^{a}_{an}) = \frac{d}{dx^{n}} (\Gamma^{a}_{ma} + \Gamma^{a}_{am}). \qquad \ldots \quad (3.2)$$

(ii) Comme m et μ sont des indices antisymétriaues, (3.1) peut aussi s'écrire comme ci-dessous

$$g^{a\lambda} \underset{n}{R^{\mu}_{am}} = g^{\mu d} \underset{m}{S^{\lambda}_{nd}}$$

en posant $\lambda = m$ et $\mu = n$, et aprés les contractions indiquées, on obtient la relation

$$g^{\lambda\mu} (E_{\lambda\mu} - H_{\lambda\mu}) = 0. \qquad \ldots \quad (3.3)$$

Pour obtenir des relations plus générales, nous commencons par un changement de notations pour les composantes des tenseurs R et S.

Comme il y a seulement six paires de (m, n), c'est-à-dire d'indices antisymmetriques, nous remplacons chaque $\binom{m}{n}$ dans les composantes par un des nombres $r = (r = 1$, ..., 6) qui est transféré sous le symbole alphabétique R ou S du tenseur.

Ainsi (3.1) prennent la forme suivante =

$$g^{a\lambda} \underset{r}{R^{\mu}_{a}} + g^{\mu d} \underset{r}{S^{\lambda}_{d}} = 0. \qquad (3.4)$$

Suivent le nombre r les 96 équations s' arrangement maintenant en six groups comportant chacun 16 relations.

Mais les $g^{a\lambda} = g^{a}_{\lambda}$, R^{μ}_{a}, S^{λ}_{d}, etc., se laissent interpréter comme les éléments de certaines matrices carrées G, R et S (les indices indiquant maintenant les lignes et les colonnes suivant des conventions bien connues), et par coséquent les relations (3.4) peuvant être représentées commue une seule équation de matrices :

$$\underset{r}{R} G + G \underset{r}{\bar{S}} = 0 \qquad (3.5)$$

où $\overline{S}$ représente la matrice transposée de S avec

$$\overline{S}^{\lambda}_{a} = S^{a}_{\lambda}.$$

En prenant les formes transposeés, on a aussi

$$\underset{r}{S}G+\overline{G}\underset{r}{\bar{R}} = 0. \tag{3.6}$$

Les épuations (3.5) et (3.6) jouissent de propriétés d'hermiticité, ainsi qu'il a été postulé par Einstein pour toutes les relations de la théoire unitaire.

En effet, on passe de (3.5) à (3.6) en transposant G et en changeant le tenseur R en S et vice versa, ce qui implique que les dites relations restent inaltérées comme un tout par les changements simultanés des $g^{\mu\nu}$ en $g^{\nu\mu}$ et des $\Gamma^{a}_{\mu\nu}$ en $\Gamma^{a}_{\nu\mu}$: exactement comme l'exige le principe d'hermiticité.

Multipliant finalement les six équations (3.6) par des nombres arbitraries λ_r, et faisant la somme, on a, aprés une transformation facile :

$$G^{-1}RG = -\bar{S} \quad (\text{où } R = \lambda_r R_r \ldots). \tag{3.7}$$

Les matrices R et $\bar{S}$ sont donc liées par une transformation qui laisse intacts leurs invariants, d'où

$$I_r(R)-(-1)^r I_r(S) = 0 \quad (r = 1, 2, 3, 4). \tag{3.8}$$

Comme les λ_r sont tout à fait arbitraries, on peut admettre que chaque coefficient de polynome en λ_r dans (3.8) pris séparément est égal à zéro.

Dans toutes les relations qui s'ènsuivent entrent seulement $\Gamma^{\lambda}_{\mu\nu}$ et leurs dérivées, chacune d'elles exprimant les propriétés caractéristiques du champ unitaire conditionneés par l'hypothése de l'existence du tenseur de potentiel.

Si l'on adopte pour les produits de la chaîne des $\underset{r}{R}$ ou des $\underset{r}{S}$ les notations suivantes :

$$\begin{gathered} \underset{r}{R}^{\lambda}_{\mu}\underset{s}{R}^{\mu}_{\lambda} = R(r, s); \quad \underset{r}{S}^{\lambda}_{\mu}\underset{s}{S}^{\mu}_{\nu}\underset{t}{S}^{\nu}_{\lambda} = S(rst), \quad \ldots ; \\ \underset{r}{R}^{\lambda}_{\lambda} = R(r), \qquad \underset{r}{S}^{\lambda}_{\lambda} = S(r), \end{gathered} \tag{3.9}$$

les relations obtenues ci-dessus pourront s'écrire aussi comme suit

$$\left.\begin{aligned} R(r) &= -S(r), \\ R(r, s) &= \quad S(rs) = S(sr), \\ R(rst) &= -S(str) = -S(rts) = -S(srt), \\ R(rstu) &= S(utsr) = S(ruts) = \ldots \end{aligned}\right\} \tag{3.10}$$

4. *a*. Comme conséauences de l'hypothése de l'existence du tenseur de potentiel, dans le champ des connexions symmétriques, on a les relations suivantes :

16 relations :

$$\text{(i)} \qquad \underset{n}{R}^{\lambda}_{lm}+\underset{l}{R}^{\lambda}_{mn}+\underset{m}{R}^{\lambda}_{nl} = 0;$$

6 relations :

(ii)
$$\underset{n}{R^{l}_{lm}} = 0;$$

45 relations :

(iii)
$$\underset{\mu}{R^{\lambda}_{ml}}\underset{\lambda}{R^{\mu}_{m'l'}} = \underset{\mu}{R^{\lambda}_{lm}}\underset{\lambda}{R^{\mu}_{l'm'}}.$$

Compte tenu des 10 équations du champ, on a en tout 77 relations homogènes entre les composantes $\Gamma^{\lambda}_{\mu\nu}$ et leurs dérivéés premières.

b. Dans le champ unitaire, on a :

6 relations :

$$R(r) = - S(r);$$

21 relations :

$$R(rs) = S(sr) ;$$

56 relations :

$$R(rst) = - S(tsr);$$

126 relations :

$$R(rstu) = S(utsr).$$

Couplées avec les équations du champ, le total peut aller jusqu'à 225 relations entre les mêmes éléments du champ.

Ainsi les conditions qui sont imposées aux éléments affines, dans la théoire unitaire, sont bien plus restrictives que celles qui sont imposées aux éléments du champ de la gravitation.

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Certain Consequences of the Existence of the Tensor g in the Affine Relativistic Field

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Summary — Relations are established where only $\Gamma^{\lambda}_{\mu\nu}$ and their derivatives enter from conditions of integrability of the equations which the tensor g of the relativistic theory satisfies.

The number of independent relations between the elements of the field is much larger in the unitary theory than in the theory of gravitation.

1. Introduction

The abstract formalism of the theory of relativity developed from the concept of the field of coefficients of affine connection of a four dimensional variety.

If δx^{μ} are the components of a parallel and infinitely small displacement to which a vector A of the chosen variety is subjected, the variations of the components A^{μ} which result are obtained by the rule

$$\delta A^{\mu} = - A^{\alpha} \Gamma^{\mu}_{\alpha\beta} \, \delta x^{\beta} \tag{1.1}$$

In the theory of gravitation the terms $\Gamma^{\mu}_{\alpha\beta}$ are symmetric, or $\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$; thus the rule of transport there is unique. On the contrary, as the coefficients are not symmetric in the unitary theory, one has here a second rule of transport :

$$\bar{\delta} A^{\mu} = -\delta x^{\alpha} \, \Gamma^{\mu}_{\alpha\beta} A^{\beta}. \tag{1.2}$$

Based on relation (1.1) we can easily calculate the tensor of curvature **R**, the components of which

$$\left.\begin{array}{l} R^{\lambda}_{\substack{\alpha\mu \\ \nu}} = \Gamma^{\lambda}_{\alpha\mu,\nu} - \Gamma^{\lambda}_{\alpha\nu,\mu} + \Gamma^{\varepsilon}_{\alpha\mu}\Gamma^{\lambda}_{\varepsilon\nu} - \Gamma^{\varepsilon}_{\alpha\nu}\Gamma^{\lambda}_{\varepsilon\mu} \\ \\ \left(\text{ou } \Gamma^{\lambda}_{\alpha u,\nu} = \frac{d}{dx^{\nu}} \Gamma^{\lambda}_{\alpha\mu} , \dots \right) \end{array}\right\} \tag{1.3}$$

are antisymmetric in the indices μ and ν ; and then by contraction we obtain the Einstein tensor **E** with the components

$$E_{\alpha\mu} = \Gamma^{\lambda}_{\alpha\mu,\lambda} - \Gamma^{\lambda}_{\alpha\lambda,\mu} + \Gamma^{\varepsilon}_{\alpha\mu}\Gamma^{\lambda}_{\varepsilon\lambda} - \Gamma^{\varepsilon}_{\alpha\lambda}\Gamma^{\lambda}_{\varepsilon\mu} \tag{1.4}$$

But the unitary theory also admits (1.2) as a rule of transport. Thus the same reasoning mentioned above will give us from (1.2) another transport S (replacing **R**) with its components

$$S^{\lambda}_{\underset{\nu}{\mu\alpha}} = \Gamma^{\lambda}_{\mu\alpha,\nu} - \Gamma^{\lambda}_{\nu\alpha,\mu} + \Gamma^{\varepsilon}_{\mu\alpha}\Gamma^{\lambda}_{\nu\varepsilon} - \Gamma^{\varepsilon}_{\nu\alpha}\Gamma^{\lambda}_{\mu\varepsilon} \tag{1.5}$$

and finally also the tensor **H** (replacing **E** of Einstein) with the components

$$H_{\mu\alpha} = \Gamma^{\lambda}_{\mu\alpha,\lambda} - \Gamma^{\lambda}_{\lambda\alpha,\mu} \mp \Gamma^{\varepsilon}_{\mu\alpha}\Gamma^{\lambda}_{\lambda\varepsilon} - \Gamma^{\varepsilon}_{\lambda\alpha}\Gamma^{\lambda}_{\mu\varepsilon}. \tag{1.6}$$

The equations $E_{\alpha\mu=0}$ are those of the field in the theory of gravitation.

The ambiguity of the law of transport in the unitary theory seems to make it difficult to choose $E_{\alpha\mu} = 0$ as fundamental equations. But the difficulty has been solved by Einstein in the following manner : it imposes four restrictive conditions on the affine field, namely

$$\Gamma^{\lambda}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\mu} = 2\Gamma_{\mu} = 0. \tag{1.7}$$

Though arbitrary, this hypothesis easily gives

$$\mathbf{E} = \mathbf{H}\ ;$$

the two laws thus lead to the same tensor. Therefore the field equations can remain the same in both the theories.

Besides the system

$$E_{\mu\nu} = 0 \tag{A}$$

which depends only on the coefficients $\Gamma^{\lambda}_{\mu\nu}$ and their first derivatives, there is also a second system where the components of the potential $g^{\mu\nu}$ and their derivatives are also included ; these are indeed the equations

$$g^{\mu\nu}_{,k} + g^{\mu\lambda}\Gamma^{\nu}_{k\lambda} + g^{\lambda\nu}\Gamma^{\mu}_{\lambda k} = 0. \tag{B}$$

We have demonstrated that (A) and (B) are obtained from one variational principle. We form the invariant integral

$$I = \int \sqrt{|g|}\, g^{\mu\nu}E_{\mu\nu}dV_4$$

with some components of the potential and the Einstein tensor (supposed to be unique due to $\Gamma_{\mu} = 0$) and we hold

$$\delta I = 0$$

for all arbitrary variations of $g^{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$; we immediately obtain the required equations.

It is evident that the existence of a potential tensor in the affine field would give it a very special character.

Indeed (B), regarded as a system of differential equations, can determine $g^{\mu\nu}$ only when the conditions of integratibility are satisfied.

Hence the relations, which follow as consequences of this hypothesis of integrability between the coefficients $\Gamma^{\lambda}_{\mu\nu}$ and their derivatives will express those relations of the characteristic properties of the field which are implicitly postulated by the theory of relativity as well as the theory of the unitary field itself.

To obtain them in an explicit form is precisely the aim of this work.

2. Case I. — When the coefficients $\Gamma^{\lambda}_{\mu\nu}$ are symmetric, the conditions of integrability of the equations (B), namely :

$$g^{\mu\nu}_{,k} + g^{\mu\lambda}\Gamma^{\nu}_{k\lambda} + g^{\lambda\nu}\Gamma^{\mu}_{\lambda k} = 0$$

can more easily be expressed in terms of the curvature tensor **R**.

If the components $\underset{n}{R^{\lambda}_{lm}}$ are expressed by the multiples $g^{\mu\nu}$ and brackets with four indices as

$$\underset{n}{R^{\lambda}_{lm}} = g^{\lambda k}\,(klmn), \tag{2.1}$$

the conditions of integrability will be well expressed by the following properties of those brackets which are well known :

$$(klmn) = -(lkmn) = (mnkl) = -(nmkl). \tag{2.2}$$

From these we simultaneously eliminate $g^{\mu\nu}$ and the brackets and obtain relations between the components of **R** only.

(i) As

$$\underset{n}{R^{\lambda}_{lm}} = g^{\lambda k}\,(klmn),$$

we hold $\lambda = l$ and contract. We now observe that the brackets are antisymmetrical in the indices (k and l) while g^{lk} are symmetrical.

We then automatically have :

$$\underset{n}{R^{l}_{lm}} = 0 \tag{2.3}$$

(ii) Similarly, since

$$\underset{n}{R^{\lambda}_{lm}}\,\underset{\lambda}{R^{n}_{l'm'}} = g^{\lambda k}\,(klmn)\,g^{nk'}\,(k'l'm'\lambda),$$

and

$$\underset{k}{R^{k'}_{ml}}\,\underset{k'}{R^{k}_{m'l'}} = g^{k'n}\,(nmlk)\,g^{\kappa\lambda}\,(\lambda\, m'l'k'),$$

we have

$$\underset{n}{R^{\lambda}_{lm}}\,\underset{\lambda}{R^{n}_{l'm'}} = \underset{k}{R^{k'}_{ml}}\,\underset{k'}{R^{k}_{m'l'}} \tag{2.4}$$

due to the relations (2.2) (with regard to the brackets).

(iii) We can write the 96 components of the tensor $\underset{m}{R^{\lambda}_{kl}}$ explicitly as functions of Γ^{λ}_{mn} and their derivatives, that is to say (1.3), which easily gives the relations

$$R^{\lambda}_{\underset{n}{lm}} + R^{\lambda}_{\underset{l}{mn}} + R^{\lambda}_{\underset{m}{nl}} = 0. \tag{2.5}$$

(2.3), (2.4), (2.5) are characteristic properties of the symmetrric field, expressed by the components of the curvature tensor **R**.

3. Case II. — When the field is non-symmetric, the conditions of integrability of the equations (B) can also be easily deduced.

With (B) we also consider the system of their first derivatives (B_1). We eliminate all the first derivatives of $g^{\mu\nu}$ between (B) and (B_1) and using the conditions

$$g^{\mu\nu}_{,l,k} - g^{\mu\nu}_{,k,l} = 0,$$

we immediately come to the following conditions of integrability :

$$0 = g^{a\lambda} R^{\mu}_{\underset{n}{am}} + g^{\mu d} S^{\lambda}_{\underset{n}{md}}, \tag{3.1}$$

from which we can immediately deduce these :

(i) By multiplying (3.1) with $g_{\mu\lambda}$ and then summing (since $g^{a\lambda} g_{\mu\lambda} = \delta^{a}_{\mu}$, $g^{\mu d} g_{\mu\lambda} = \delta^{d}_{\lambda}$ and δ 's are the symbols of Kronecker) we get

$$R^{a}_{\underset{n}{am}} + S^{d}_{\underset{n}{md}} = 0$$

or

$$\frac{d}{dx^{m}} (\Gamma^{a}_{na} + \Gamma^{a}_{an}) = \frac{d}{dx^{n}} (\Gamma^{a}_{ma} + \Gamma^{a}_{am}). \tag{3.2}$$

(ii) As m and μ are the antisymmetric indices, (3.1) can also be written as

$$g^{a\lambda} R^{\mu}_{\underset{n}{am}} = g^{\mu d} S^{\lambda}_{\underset{m}{nd}}$$

by putting $\lambda = m$ and $\mu = n$, and then after the contraction indicated, we obtain the relation

$$g^{\lambda\mu} (E_{\lambda\mu} - H_{\lambda\mu}) = 0. \tag{3.3}$$

In order to obtain more general relations, we begin with a change of notations for the components of the tensors R and S.

As there are only six pairs of (m, n), i.e. antisymmetric indices, each $\binom{m}{n}$ in the component is replaced by one of the numbers r ($r=1, \ldots, 6$) which replaces the symbol **R** or **S** of the tensor.

Thus (3.1) take the following form =

$$g^{a\lambda} \underset{r}{R^{\mu}_{a}} + g^{\mu d} \underset{r}{S^{\lambda}_{d}} = 0. \tag{3.4}$$

Following the numbers r, the 96 equations arrange themselves into six groups each having 16 relations.

But $g^{a\lambda} = g^{a}_{\lambda}$, R^{μ}_{a} , S^{λ}_{d}, etc can be interpretated as the elements of certain square matrices G, R and S (the indices now indicate the rows and columns as per the well known conventions), and consequently the relations (3.4) can be represented as a single equation of the matrices :

$$\underset{r}{R}G + G\underset{r}{\bar{S}} = 0 \tag{3.5}$$

where $\mathbf{\bar{S}}$ represents the transposed matrix of $\mathbf{S}$ with

$$\bar{S}^{\lambda}_{a} = S^{a}_{\lambda}.$$

By considering the transposed forms, we also have

$$\underset{r}{S}G + \bar{G}\underset{r}{\bar{R}} = 0. \tag{3.6}$$

The equations (3.5) and (3.6) have the properties of hermiticity as it was postulated by Einstein for all the relations of the unitary theory.

In fact, we pass from (3.5) to (3.6) by transposing G and changing the tensor $\mathbf{R}$ and $\mathbf{S}$ and vice versa, which implies that the said relations remain unaltered as a whole by the simultaneous changes of $g^{\mu\nu}$ into $g^{\nu\mu}$ and of $\Gamma^{a}_{\mu\nu}$ into $\Gamma^{a}_{\nu\mu}$, exactly as per the principle of hermiticity.

Finally, by multiplying the six equations (3.6) by arbitrary numbers λ_r and summing up, we get after an easy transformation :

$$G^{-1} R\, G = -\bar{S} \quad (\text{where } R = \lambda_r \underset{r}{R} \cdots). \tag{3.7}$$

The matrices R and $\bar{S}$ are therefore linked by a transformation which leaves their invariants intact, from which

$$I_r(R) - (-1)^r I_r(S) = 0 \quad (r = 1, 2, 3, 4). \tag{3.8}$$

As the λ_r are completely arbitrary, it can be said that each coefficient of the polynomial in λ_r in (3.8), taken separately, is equal to zero.

In all the relations that arise only $\Gamma^{\lambda}_{\mu\nu}$ and their derivatives enter, each of them expressing the characteristic properties of the unitary field conditioned by the hypothesis of the existence of the potential tensor.

If the products of the chain of the $\underset{r}{R}$ or the $\underset{r}{S}$ are given the following notations :

$$\underset{r}{R}{}^{\lambda}_{\mu}\underset{s}{R}{}^{\mu}_{\lambda} = R(r, s); \quad \underset{r}{S}{}^{\lambda}_{\mu}\underset{s}{S}{}^{\mu}_{\nu}\underset{t}{S}{}^{\nu}_{\lambda} = S(rst), \quad \ldots ; \tag{3.9}$$

$$\underset{r}{R}{}^{\lambda}_{\lambda} = R(r), \qquad \underset{r}{S}{}^{\lambda}_{\lambda} = S(r),$$

the relations obtained above can also be written as

$$\left.\begin{aligned} &R(r) = -S(r), \\ &R(r, s) = \quad S(rs) = S(sr), \\ &\cdot R(rst) = -S(str) = -S(rts) = -S(srt), \\ &R(rstu) = S(utsr) = S(ruts) = \ldots \end{aligned}\right\} \tag{3.10}$$

4.a. As consequences of the hypothesis of the existence of the potential tensor in the field of symmetric connections, the following relations are obtained :

16 relations :

(i)
$$\underset{n}{R^{\lambda}_{lm}} + \underset{l}{R^{\lambda}_{mn}} + \underset{m}{R^{\lambda}_{nl}} = 0;$$

6 relations :

(ii)
$$\underset{n}{R^{l}_{lm}} = 0;$$

45 relations :

(iii)
$$\underset{\mu}{R^{\lambda}_{ml}} \underset{\lambda}{R^{\mu}_{m'l'}} = \underset{\mu}{R^{\lambda}_{lm}} \underset{\lambda}{R^{\mu}_{l'm'}}.$$

Taking into account the 10 field equations, we have a total of 77 homogenous relations between the components $\Gamma^{\lambda}_{\mu\nu}$ and their first derivatives.

b. In the unitary field, we have :

6 relations :

$$R(r) = -S(r);$$

21 relations :

$$R(rs) = S(sr)\ ;$$

56 relations :

$$R(rst) = -S(tsr);$$

126 relations :

$$R(rstu) = S(utsr).$$

Coupled with the field equations the total can go upto 225 relations between the same elements of the field.

Thus the conditions which are imposed on the affine elements in the unitary theory are more restrictive than those that are imposed on the elements of the gravitational field.

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THE AFFINE CONNECTION IN EINSTEIN'S NEW UNITARY FIELD THEORY

By S. N. Bose

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The non-symmetric $\Gamma^a_{\mu\nu}$'s are defined by the 64 equations

$$\frac{\partial g_{\mu\nu}}{\partial x_\sigma} = g_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} + g_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} \tag{1}$$

in the new theory. A method of obtaining explicit expressions of Γ's in terms of $g_{\mu\nu}$ and its derivatives is sketched in this note. Following the usual convention of the relativity theory, the summation symbols are suppressed throughout, it being understood that summation is always implied whenever dummy indices occur.

I. The generalized Christoffel bracket $\binom{\mu\nu}{\sigma}$ is defined by the equation

$$\binom{\mu\nu}{\sigma} = \tfrac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\sigma\nu}}{\partial x_\mu} - \frac{\partial g_{\nu\mu}}{\partial x_\sigma}\right); \tag{2}$$

(the subscripts are written in a cyclic order (Schrödinger)). Also,

$$\binom{\mu\nu}{\sigma} = s_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} + a_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} + a_{\lambda\nu}\Gamma^\lambda_{\sigma\mu} \tag{3}$$

where s, a, denote symmetric and anti-symmetric components of g. If $\Gamma^a_{\mu\nu}$ is similarly decomposed into

$$\Sigma^a_{\mu\nu} = \tfrac{1}{2}(\Gamma^a_{\mu\nu} + \Gamma^a_{\nu\mu})$$

(4) and

$$V^a_{\mu\nu} = \tfrac{1}{2}(\Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu}),$$

and the bracket $\binom{\mu\nu}{\sigma}$ into

$$\genfrac{[}{]}{0pt}{}{\mu\nu}{\sigma} = \tfrac{1}{2}\left[\binom{\mu\nu}{\sigma} + \binom{\nu\mu}{\sigma}\right]$$

(5) and

$$\genfrac{\{}{\}}{0pt}{}{\mu\nu}{\sigma} = \tfrac{1}{2}\cdot\left[\binom{\mu\nu}{\sigma} - \binom{\nu\mu}{\sigma}\right]$$

the following relations follow immediately

$$\text{(A)} \qquad \begin{cases} s_{\sigma\lambda}\Sigma^{\lambda}_{\mu\nu} + a_{\mu\lambda}V^{\lambda}_{\nu\sigma} + a_{\lambda\nu}V^{\lambda}_{\sigma\mu} = \genfrac{[}{]}{0pt}{}{\mu\nu}{\sigma} \\ s_{\sigma\lambda}V^{\lambda}_{\mu\nu} + a_{\mu\lambda}\Sigma^{\lambda}_{\nu\sigma} + a_{\lambda\nu}\Sigma^{\lambda}_{\sigma\mu} = \genfrac{\{}{\}}{0pt}{}{\mu\nu}{\sigma}. \end{cases}$$

Again if

$$\text{(6)} \qquad \begin{aligned} s_{\sigma\lambda}\Sigma^{\lambda}_{\mu\nu} &= P_{\sigma\mu\nu}, \text{ or } \Sigma^{\lambda}_{\mu\nu} = s^{\lambda k}P_{k\mu\nu} \\ s_{\sigma\lambda}V^{\lambda}_{\mu\nu} &= T_{\sigma\mu\nu}\text{ , or } V^{\lambda}_{\mu\nu} = s^{\lambda k}T_{k\mu\nu}\text{ ,} \end{aligned}$$

we can express Σ and V in terms of P and T with the help of $s^{\mu\nu}$, the contravariant component of $s_{\mu\nu}$ obtained in the usual manner; also taking

$$C^{\sigma}_{\mu} = s^{\sigma\lambda}a_{\lambda\mu}$$

as the components of the matrix product of s and a, we can rewrite the equations in the form

$$\text{(A}'\text{)} \qquad \begin{aligned} \genfrac{[}{]}{0pt}{}{\mu\nu}{\sigma} &= P_{\sigma\mu\nu} - C^{k}_{\mu}T_{k\nu\sigma} + C^{k}_{\nu}T_{k\sigma\mu} \\ \genfrac{\{}{\}}{0pt}{}{\mu\nu}{\sigma} &= T_{\sigma\mu\nu} - C^{k}_{\mu}P_{k\nu\sigma} + C^{k}_{\nu}P_{k\sigma\mu}\,. \end{aligned}$$

P's can now be easily eliminated, and a tensor equation, involving only T's is obtained in the form

$$\text{(B)} \qquad \begin{aligned} &\genfrac{\{}{\}}{0pt}{}{\mu\nu}{\sigma} + C^{k}_{\mu}\genfrac{[}{]}{0pt}{}{\nu\sigma}{k} - C^{k}_{\nu}\genfrac{[}{]}{0pt}{}{\sigma\mu}{k} \\ &\qquad = T_{\sigma\mu\nu} + C^{\lambda}_{\sigma}(T_{\lambda k\nu}C^{k}_{\mu} + T_{\lambda\mu k}C^{k}_{\nu}) - C^{\lambda}_{\mu}C^{k}_{\nu}(T_{\lambda k\sigma} + T_{k\sigma\lambda}). \end{aligned}$$

This tensor equation may be regarded as characterizing the continuum of the Unitary Theory.

When (B) is solved, yielding T in terms of known functions of g and its derivatives, and also C^{λ}_{k}'s, the substitution in (A) of T immediately gives the P's and hence also the Γ's.

To solve this central equation we first observe that a cyclic addition easily yields

$$\text{(7)} \qquad \begin{aligned} T_{\sigma\mu\nu} + T_{\mu\nu\sigma} + T_{\nu\sigma\mu} = T(\mu\nu\sigma) &= \genfrac{\{}{\}}{0pt}{}{\mu\nu}{\sigma} + \genfrac{\{}{\}}{0pt}{}{\nu\sigma}{\mu} + \genfrac{\{}{\}}{0pt}{}{\sigma\mu}{\nu} = \{\sigma\mu\nu\} \\ \{\sigma\mu\nu\} &= \tfrac{1}{2}\left(\frac{\partial a_{\mu\sigma}}{\partial x_{\nu}} + \frac{\partial a_{\sigma\nu}}{\partial x_{\mu}} + \frac{\partial a_{\nu\mu}}{\partial x_{\sigma}}\right) \end{aligned}$$

which is easily recognized as a tensor.

Also if Christoffel co-efficients $\gamma^{a}_{\mu\nu} = s^{at}\genfrac{[}{]}{0pt}{}{\mu\nu}{t}$ (the usual symmetric γ's) are used, the left-hand side of (B) is changed to

$$\{\mu\nu\sigma\} + a_{\mu\nu\dot{\sigma}}$$

where

$$a_{\mu\nu\sigma} = \frac{\partial a_{\mu\nu}}{\partial x_\sigma} - a_{\mu\lambda}\gamma^{\lambda}_{\sigma\nu} - a_{\lambda\nu}\gamma^{\lambda}_{\mu\sigma} \tag{8}$$

is the familiar covariant derivative with symmetric $\gamma^{a}_{\mu\nu}$'s as in Riemann-Geometry; also observing that

$$T_{\lambda k\sigma} + T_{k\sigma\lambda} = T(\sigma\lambda k) - T_{\sigma\lambda k} = \{\sigma\lambda k\} - T_{\sigma\lambda k}\,, \tag{9}$$

the equation (B) can be transformed into the form

$$\text{(C)}\quad \begin{cases} \{\sigma\mu\nu\} + \{\sigma k\lambda\}C^{k}_{\mu}C^{\lambda}_{\nu} + a_{\mu\nu\sigma} \equiv U_{\sigma\mu\nu} \\ U_{\sigma\mu\nu} \equiv T_{\sigma\mu\nu} + T_{\sigma k\lambda}C^{k}_{\mu}C^{\lambda}_{\nu} + C^{\lambda}_{\sigma}[T_{\lambda k\nu}C^{k}_{\mu} + T_{\lambda\mu k}C^{k}_{\nu}] \end{cases}$$

where the anti-symmetric suffixes are put in the end on both sides in the tensors with three-index symbols.

II. To solve the equation (C) we remark that the co-efficients C^{k}_{μ} that occur in the equation are the components of the matrix

$$C = sa = \| C^{k}_{\mu} \| \text{ or } C^{k}_{\mu} = s^{kt} a_{t\mu}\,.$$

The general equation for the eigen vector M of the matrix C can be written in the form

$$| C - pE \mid M = 0 \text{ or } C^{k}_{\mu} M^{\mu}_{p} = p M^{k}_{p}\,; \tag{10}$$

in the vector $M = M^{\sigma}_{p}$ the superscript σ indicates the component, and the subscript p the associated eigen-value.

The characteristic eigen values are the roots of the quartic

$$x^4 + I_2 x^2 + I_4 = 0 \tag{11}$$

as the first and third invariants vanish (the matrix being the product of a symmetric matrix).

A few well-known properties of matrix equations are here re- capitulated for ready reference.

1.A matrix C, and its transposition $\overline{C}$ have the same eigen values, but different eigen vectors ; and the eigen vectors, written with two suffixes, (representing the component and the eigen value) satisfy the following relations :

$$\overline{M}^{\sigma}_{k} M^{\sigma}_{k'} = \delta^{k}_{k'}\,, \quad \overline{M}^{\sigma}_{k} M^{\sigma'}_{k} = \delta^{\sigma}_{\sigma'}\,.$$

1 (a). The theorem is true even in the case of repeated roots of the characteristic equation of C, if the eigen vectors are suitably constructed.

2. Taking the elements of the matrix C, as C^k_μ, where k denotes the rows and μ the columns of the matrix, a new compound matrix $C \times C$, can be constructed in the following manner :

$$(C \times C)\binom{kl}{\mu\nu} = C^k_\mu C^l_y - C^k_\nu C^l_\mu \tag{12}$$

obtained by the usual cross-multiplication rule. The elements can be arranged in a six × six matrix in any proper order ; e.g.,

(23) → 1, (31) → 2, (12) → 3, (14) → 4, (24) → 5, (34) → 6.

3. The eigen vectors of the matrixes $C \times C$ (arranged in any assumed manner) may be obtained by taking the vector product of the eigen vectors M^σ_p of C. $M_a \times M_b$ is thus the eigen six-vector of the six matrix, $C \times C$ corresponding to the eigen value $p_a p_b$.

4. The eigen vector of $\bar{C} \times \bar{C}$ is similarly obtained as $\bar{M}_a \times \bar{M}_b$ corresponding to the same eigen value $p_a p_b$.

5. The eigen vectors of the compound matrices have corresponding orthogonal properties.

The components of the eigen vectors can be written by the notation used before as $M\binom{\sigma\sigma'}{ab}$ and $\bar{M}\binom{\epsilon\epsilon'}{cd}$.

The orthogonality relation is written as

$$\begin{cases} \bar{M}\binom{\sigma\sigma'}{ab} M\binom{\sigma\sigma'}{cd} = \delta^a_c \delta^b_d \\ \bar{M}\binom{\sigma\sigma'}{ab} M\binom{\epsilon\epsilon'}{ab} = \delta^\sigma_\epsilon \delta^{\sigma'}_{\epsilon'} . \end{cases} \tag{13}$$

III. Keeping the above well-known properties of matrices in mind we can tackle the problem of solving the general equation. We multiply the equation (C') by $M^\sigma_a M^\mu_b M^\nu_c$ formed from the eigen vectors of C, corresponding to three eigen values a, b, and c, where $b \neq c$; then remembering the equations

$$C^k_\mu M^\mu_b = p_b M^k_l , \qquad C^k_\nu M^\nu_c = p_c M^k_c \text{, etc.,} \qquad (b, c \text{ not summed})$$

we get easily, by suitably changing the dummies

$$\begin{cases} M^\sigma_a M^\mu_b M^\nu_c U_{\sigma\mu\nu} \equiv \{hkl\} M^h_a M^k_b M^l_c (1 + p_b p_c) + a_{klh} M^h_a M^k_b M^l_c ; \\ \text{also} \\ M^\sigma_a M^\mu_b M^\nu_c U_{\sigma\mu\nu} = T_{hkl} M^h_a M^k_b M^l_c (1 + p_a p_b + p_b p_c + p_c p_a). \end{cases} \tag{14}$$

Or remembering the anti-symmetric nature of the suffixes k, l we write the same equation with k, l in the reversed order and add, using the symbol

$$M\binom{kl}{bc} = M^k_a M^l_c - M^k_c M^l_b$$

easily getting the result in the form

$$(15)\qquad (1 + p_a p_b + p_b p_c + p_c p_a)^{-1}[(1 + p_b p_c)\{hkl\} M^h_a M\binom{kl}{bc} + a_{kch} M^h_a M\binom{kl}{bc}] = T_{hkl} M^h_a M\binom{kl}{bc}.$$

Remembering the orthogonal properties of the eigen vectors, namely equations (12) and (13), and multiplying by $\bar{M}^\sigma_a \bar{M}\binom{\mu\nu}{bc}$ on both sides followed by summation on a, b, c, we have

$$(16)\qquad T_{\sigma\mu\nu} = (1 + p_a p_b + p_b p_c + p_c p_a)^{-1}[(1 + p_b p_c)\{hkl\} \bar{M}^\sigma_a \bar{M}\binom{\mu\nu}{bc} M^h_a M\binom{kl}{bc} + a_{klh} \bar{M}^\sigma_a \bar{M}\binom{\mu\nu}{bc} M^h_a M\binom{kl}{bc}].$$

An explicit form of $T_{\sigma\mu\nu}$ is thus obtained in terms of the eigen vectors of C and $\bar{C}$ and the covariant derivatives of a_{kl} calculated with the symmetric Christoffel co-efficients, γ's obtained from the symmetric parts of $g_{\mu\nu}$, namely $s_{\mu\nu}$. The discriminant-determinant of the equation (C) is obviously

$$(17)\qquad D_{24} = \prod (1 + p_a p_b + p_b p_c + p_c p_a) (\text{product with roots of (11) with } b \neq c).$$

The value of the determinant can be easily calculated in terms of the invariants of C; in fact,

$$(18)\qquad D_{24} \equiv (1 + I_2 + I_4)^8 [(1 + 5I_4 - I_2)^2 - 4I_4(2 - I_4)^2]^2.$$

The determinant D always vanish when $1 + I_2 + I_4 = 0$; but this happens when the determinant of $g = \| g_{\mu\nu} \| = 0$.

This is proved as follows:

Any transformation $s(\quad)s^{-1}$, where $s^{-1} = \bar{s}$, preserves the symmetric or/and the anti-symmetric nature of a matrix; $s_{\mu\nu}$ part of $g_{\mu\nu}$ may be diagonalized, and at the same time the anti-symmetric tensor $a_{\mu\nu}$ preserves its anti-symmetric character. Hence g can be reduced to the form

$$\begin{vmatrix} s_{11} & a_{12} & \cdots \\ a_{41} & \cdots & s_{44} \end{vmatrix} \text{ and } a_{\mu\nu} + a_{\nu\mu} = 0$$

by such a transformation.

The values of all invariants remain obviously unaltered. In this special co-ordinates

$$\Delta = \| g_{\mu\nu} \| = s_{11}s_{22}s_{33}s_{44} + \Sigma s_{11}s_{22}a_{34}^2 + (a_{12}a_{34} + \cdots).^2 \tag{19}$$

The invariants of the matrix C can be easily calculated which gives

$$I_2 = (s_{11}s_{22}s_{33}s_{44})^{-1}\Sigma s_{11}s_{22}a_{34}^2$$

$$I_4 = (s_{11}s_{22}s_{33}s_{44})^{-1}(a_{12}a_{34} + \cdots)^2$$

hence

$$\Delta = s_1s_2s_3s_4(1 + I_2 + I_4). \tag{20}$$

Hence if $(1 + I_2 + I_4) = 0$

$$\| g_{\mu\nu} \| = 0 \text{ also.}$$

That when T is determined Γ's are also determined follow simply as a consequence of equation (A).

IV. In later developments of his theory Einstein has introduced the additional condition

$$\Gamma_l \equiv \Gamma^\lambda_{l\lambda} - \Gamma^\lambda_{\lambda l} = 0;$$

however, from the contravariant form of equation (1), viz.:

$$-\frac{\partial g^{\mu\nu}}{\partial x_\sigma} = -g^{\mu\nu}_\sigma = g^{\mu\lambda}\Gamma^\nu_{\sigma\lambda} + g^{\lambda\nu}\Gamma^\mu_{\lambda\sigma} \tag{21}$$

the following relation

$$[| g |^{\frac{1}{2}}(g^{\mu\lambda} - g^{\nu\mu})]_\mu = \tfrac{1}{2} | g |^{\frac{1}{2}}(g^{\nu\lambda} + g^{\lambda\nu})(\Gamma^\mu_{\lambda\mu} - \Gamma^\mu_{\mu\nu}) \tag{22}$$

follows easily. The condition

$$\Gamma^\lambda_{\lambda l} - \Gamma^\lambda_{\lambda l} = 0$$

means four relations involving the differential co-efficients of $a_{\mu\nu}$ and $g_{\mu\nu}$. In the equation (C) for the determination of T, all first order differential co-efficients occur in the left side of the in-homogeneous equation, whereas the homogeneous part involves only C^{λ}_{k}, composed of $s^{\mu\nu}$ and $a_{\mu\nu}$ and no differential co-efficients occur here. Hence the additional conditions do not affect the general nature of the solution.

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XVIII. A REPORT ON THE STUDY OF THERMOLUMINESCENCE

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(*Received for publication on February 9, 1955*)

When some solids are irradiated with ionizing radiations like X-ray or cathode rays, they often exhibit fluorescence followed by a long period phosphorescent afterglow. Again, if the irradiated sample is heated, the stored energy is released in the form of radiation, emitted at various temperatures characteristic of the sample. This thermo-stimulated release of energy is commonly known as thermoluminescence. The capacity of substances to store energy is large at low temperatures, so thermoluminescence is pronounced if the irradiation is carried out at low temperature (*e.g.*, at −184°C). We may plot in a thermoluminescence curve the total intensity obtained from the sample at various temperatures during heating. From the curve, the trap depths of light-storing states below the conduction band, can be evaluated[1]. Shallow traps yield glow peaks at comparatively low temperatures, while deeper traps are released at higher temperatures. Quartz, alkali halides, calcite, glasses and many other substances show this kind of thermoluminescence. Of late, thermoluminescence has been a useful tool in many problems of research. Dosimetry of radiations, study of heterogeneous catalysts, ionic nature of elements in glasses, and identification of clay minerals are some of the uses to which thermoluminescence has been put.

Study of thermoluminescence of alkali halides was initiated in this laboratory in 1950, as a part of a larger programme of investigating solid state and of co-ordinating fluorescence study with the results of soft X-ray spectroscopy. The fluorescence spectra of alkali halides both at room and low temperatures had already been studied[2, 3, 4]. Measurements of the afterglow[5] decay rates had also been done, so thermoluminescence study of alkali halides and also of some organic substance was undertaken, with a view to clarify the processes involved during energy storage in these phosphors.

In this laboratory, thermoluminescence has been studied by exciting the sample at liquid oxygen temperature, in a demountable cathode-ray tube fitted with a window of quartz. The sample is mounted as a fine coating on a hollow bulb made of thin silver. The bulb can be filled with liquid oxygen to keep the sample at low temperature and it may be rapidly heated with an electric heater immersed in the bath. A thermocouple on the surface of the bulb, connected to a Moll galvanometer, gives temperature of the sample. Another galvanometer connected to the photomultplier tube, placed facing the specimen, gives luminescence intensity. The movements of the two galvanometers during thermoluminescence are recorded on a slow rotating photographic drum. A typical example of thermoluminescence curve is shown in Fig. 1.

For recording thermoluminescence, different photomultiplier tubes (931A, 1P28, 1P22) have been employed to suit the emissions of the different samples. Fluorescence spectra of the substance being known from a previous study, there was no difficulty in selecting the photomultipliers. Although low rate of heating (0·25°C/sec.) is usually preferred, yet it has been observed that, in the case of many phosphors, the weaker peaks get flattened out unless the rate of heating is high (6–10°C/sec.). A high rate of heating

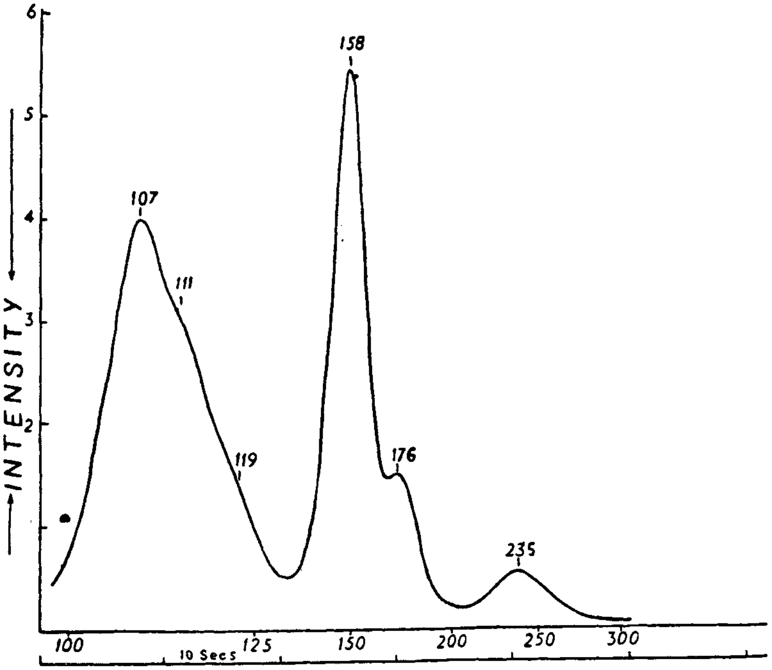

FIG. 1. Thermoluminescence of KI.

was often maintained and a fine film of powdered sample was found more suitable than a single crystal. For recording the spectral nature of the different glow peaks, two photomultipliers, sensitive in different regions, were made to record thermoluminescence simultaneously during the same run of the experiment, and optical filters were used to separate out the emissions.

Quite a number of alkali halides, both pure and activated, has been studied and some of the results have been published[6, 7]. These halides yield a number of thermoluminescence peaks between 90°K and 600°K. It has been possible to explain some of the results of afterglow decay rates in the light of information obtained from thermoluminescence study. The effects of thallium activation of samples on thermoluminescence have been investigated and it has been found that in some samples thallium produces a new peak of its own, without affecting the peaks of the parent lattice. But in some samples, like KCl and KI, it completely changes the thermoluminescence curve. From the comparative method of study with optical filters, it has been found that the different peaks emitted by a specimen are not spectrally identical. In general the peaks emitted at high temperatures are rich in emissions of ultra-violet and blue regions. But thorough study of any peak with an ordinary spectrograph is not possible, because the individual peaks disappear within a few seconds. To study the spectral nature of individual peaks, construction of a rapid spectrophotometer was deemed necessary and has been recently completed. With the help of this spectrophotometer some interesting results have been recently obtained (see Figs. 2, 3, 4 and 5).

The automatic rapid scanning spectrophotometer employs two concave mirrors of 25 cm. focal length and large focal ratios of *f*/4·2 constructed from stainless steel and a large prism of quartz. Light from the first slit is collimated by the first concave mirror, and, after dispersion by the prism, the beam is focussed by means of the other concave mirror. But before falling on the exit slit, the spectrum is reflected by another plane mirror. The latter is mounted on a turntable, which can be made to oscillate to and fro with the help of an induction motor and a cam. The entire spectrum (ultra-violet and visible) sweeps before the slit in about 0·95 second and flies back instantaneously to the original position by a cam arrangement. Just behind the exit slit is placed the photomultiplier tube which receives the light. The output of the photomultiplier tube passing through a pre-amplifier is amplified and displayed on a Dumont 304H cathode-ray tube with long time. A micro-switch attached to the cam triggers the sweep circuit every time the ultra-violet end of the spectrum just comes near the slit so that the same part of the spectrum is given at the same position. During each sweep, a specially adopted synchronous 35 mm. camera arrangement photographs the intensity distribution curve displayed on the cathode-ray screen. With the help of this spectrophotometer, the spectral intensity distribution of the emission can be photographed in less than one second and the apparatus is capable of giving all the necessary information during the entire thermoluminescence experiment. Not only the spectral composition and the intensity of the different peaks can be obtained, but also the changes in emissions in subsequent periods of a single thermoluminescence can be easily and accurately studied. Commercial types of photomultipliers were found to be not sensitive enough for this work, so a special photomultiplier with 19 dynodes was procured from France.

It is well-known that most of the solids (like alkali halides, quartz, calcite), on irradiation with ionizing radiations, develop colour centres. From simultaneous changes in colour centres and by recording diffuse reflectivity during thermoluminescence, it has been shown that some of the luminescence peaks correspond with thermally actuated transformation in colour centres. It has been found that thermoluminescence-behaviour of alklali halides is very susceptible to previous history of heat treatment of the sample. Other properties have also been studied in this laboratory.

Thermoluminescence of some aromatic hydrocarbons both solids and liquids between 90°K and 250°K has also been investigated. This work gives information regarding the metastable states of the molecules excited by cathode-rays at low temperatures. The depth of the metastable states below the upper excited states, as measured from the thermoluminescence, agrees fairly with that obtained from the study of fluorescence and phosphorescence spectra. Thermoluminescence opens a new method of investigation of the metastable states of the molecules.

This article would not be complete unless another interesting application of thermoluminescence is mentioned. We know that many naturally occurring minerals exhibit thermoluminescence. Recently a number of samples of sands procured from various parts of this country was studied. All sands show feeble thermoluminescence, but those from Bargarh and Mangalhat showed pronounced one. Measurement of γ-ray counts in the case of the latter showed slightly extra counts over the background. This leads to the possibility that thermoluminescence might be developed as a preliminary method of detecting radioactive minerals. Many minerals (like bentonite) are often found to yield thermoluminescent radiation. The ultimate reason for the occurrence of thermoluminescence in natural minerals has not yet been found. Radioactivity may play some part in this occurrence, but it is yet too early to make any definite statement on this matter.

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Bose, Sharma and Dutta **Trans. Bose Inst., Com. Vol., Plate I.**

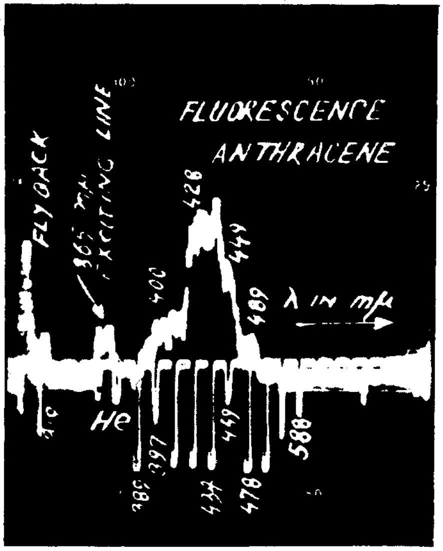

FIG. 2. Shows the fluorescent spectra of anthracene.

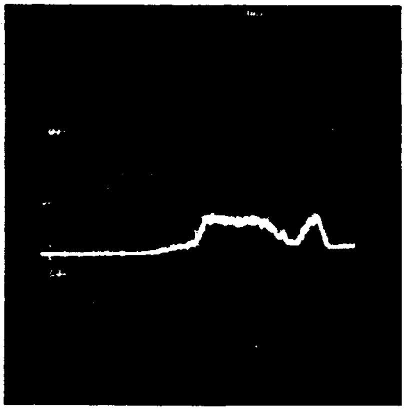

FIG. 3.

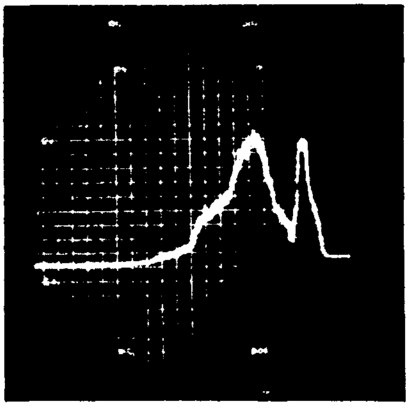

FIG. 4.

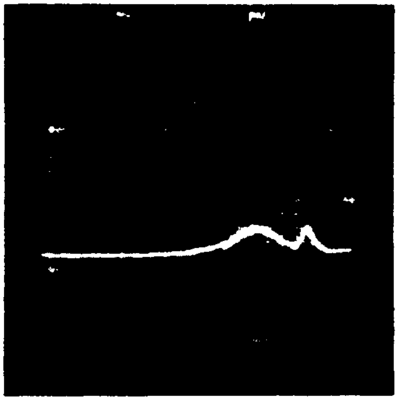

FIG. 5.

FIGS. 3, 4 and 5 give the radiation as observed from a sample of irradiated NaCl during heating. (Note change of character with time.) Figs. 3, 4 and 5 are photographs taken at 3, 7 and 17 secs. respectively after the commencement of heating.

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SOLUTION D'UNE ÉQUATION TENSORIELLE INTERVENANT DANS LA THÉORIE DU CHAMP UNITAIRE;

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Résumé.—On exprime explicitement en fonction des C et de leurs invariants la solution de l'équation tensorielle d'où dépend la determination des $\Gamma^{\lambda}_{\mu\nu}$.

On remarque que l'itération fournit une solution rigoureuse quand det C = o

1. Dans un précédent article (1), on a montré qu'il est possible de ramener le calcul des coefficients affines $\Gamma^{\lambda}_{\mu\nu}$ en fonction des $g_{\mu\nu}$ et de leurs dérivées à la solution de l'équation tensorielle

$$U_{a\mu\nu} = T_{a\mu\nu} + T_{akl}C^{k}_{\mu}C^{l}_{\nu} + C^{l}_{a}(T_{lk\nu}C^{k}_{\mu} + T_{l\mu l}C^{l}_{\nu}). \tag{1.1}$$

Les tenseurs U et T sont antisymétriques par rapport aux indices μ et ν, C^{k}_{μ} sont les éléments d'une matrice C formée à partir des parties symmétriques et antisymétriques du tenseur g :

$$\left\{\begin{array}{l} C = Sa, \quad a_{\mu\nu} = \dfrac{1}{2}(g_{\mu\nu} - g_{\nu\mu}), \quad s_{tk} = \dfrac{1}{2}(g_{tk} + g_{kt}), \\ \qquad\qquad\qquad S^{\mu t}S_{tk} = \delta^{\mu}_{k} \end{array}\right. \tag{1.2}$$

Les valeurs propers x de l matrice C vérifient l'équation

$$x^4 + I_2x^2 + I_4 = 0, \tag{1.3}$$

I_2 et I_4 sont les invariants d'ordre pair de C; de l'équation (1.2) il résulte immédiatement que les invariants impairs sont nuls.

Le vecteur propre M^{k}_{a} de C correspondant à la valeur propre a vérifie l'équation

$$C^{\mu}_{k}M^{k}_{a} = aM^{\mu}_{a}. \tag{1.4}$$

La matrice transposée $\overline{C}(\overline{C}^{\mu}_{k} = C^{k}_{\mu})$ a les mêmes valeurs propres que C mais ses vecteurs propres $\overline{M}^{k}_{a}$ sont différents.

Si l'équation caractéristique a toutes ses racines distinctes, il existe une série complète de vecteurs propres M et $\overline{M}$ de C et $\overline{C}$ et l'on peut exprimer les éléments de C en fonction des vecteurs propres : Savoir

$$C^{k}_{\mu} = \Sigma\,\lambda\overline{M}^{\mu}_{\lambda}M^{k}_{\lambda}. \tag{1.5}$$

On calcule facilement la solution de (1.1) :

$$(1.6)\quad \begin{cases} T_{\alpha\mu\nu} = \sum\limits_{abc} (1+ab+bc+ca)^{-1} \overline{M}_a^{\sigma} M_a^{x} \overline{M}\begin{pmatrix}\mu\nu\\ bc\end{pmatrix} M\begin{pmatrix}yz\\ bc\end{pmatrix} U_{xyz} \quad (b \neq c), \\ \\ M\begin{pmatrix}yz\\ bc\end{pmatrix} = M_b^y M_c^z - M_c^y M_b^z\ ; \end{cases}$$

et la sommation porte sur les 24 combinaisons possibles des (abc).

On peut écrire la solution (1.6) sous la forme d'une équation tensorielle

$$T = BU,$$

le tenseur B ayant pour composantes :

$$(1.7)\qquad B_{\sigma\mu\nu}^{xyz} = \Sigma\, \frac{\overline{M}_a^{\sigma} \overline{M}\begin{pmatrix}\mu\nu\\ bc\end{pmatrix} M_a^x M\begin{pmatrix}yz\\ bc\end{pmatrix}}{1+bc+ca+ab}$$

(1.7) est identique à (1.5).

Je me propose de faire la sommation dans (1.7) et d'obtenir les composantes de B en fonction des éléments de C.

2. Au moyen de C et de la matrice unitaire E nous pouvons former une algèbre $\{C_4\}$ avec addition et multiplication matricielle.

La formule générale d'une matrice A dans $\{C_4\}$ est

$$(2.1)\qquad \begin{cases} A = \lambda_0 + \mu_1 C_1 + \mu_2 C_2 + \mu_3 C_3; \\ C_1 = C, \quad C_2 = CC, \quad C_3 = CCC. \end{cases}$$

Toutes les matrices A commutent et ont donc les mêmes vecteurs propres M que C.

Les matrices $\overline{A}$ de l'algèbre transposée ont aussi les mêmes vecteurs propres $\overline{M}$ que $\overline{C}$.

Si $a, \overline{a}, b, \overline{b}$ sont les racines de l'équation caractéristique (1.3), on voit facilement que les matrices de base se développent ainsi :

$$(2.2)\quad \begin{cases} E = f(a)+f(\bar{a})+f(b)+f(\bar{b}): & C_2 = a^2[f(a)+f(\bar{a})]+b^2[f(b)+f(\bar{b})]; \\ C_1 = a[f(a)-f(\bar{a})+b[f(b)+f(\bar{b})]; & C_3 = a^3[(fa)-f(\bar{a})]+b^3[f(b)-f(\bar{b})]; \end{cases}$$

où $f(a) = \overline{M}_a M_a, \ldots$ en ajoutant les indices inférieurs et supérieurs aux formules ci-dessus, onobtient

$$C_\mu^\lambda = \left(\overline{M}_a^\mu M_a^\lambda - \overline{M}_{\bar{a}}^\mu M_{\bar{a}}^\lambda \right) a + b \left(\overline{M}_b^\mu M_b^\lambda - \overline{M}_{\bar{b}}^\mu M_{\bar{b}}^\lambda \right)$$

Déduisons maintenant une nouvelle algèbre $\{V_6\}$: où l'on forme ainsi les matrices V à partir de deux éléments quelconques A, B, de $\{C_4\}$:

$$V = (AB),$$

avec

$$V^{\mu\nu}_{rs} = \frac{1}{2}\left(A^\mu_r B^\nu_s + A^\nu_s B^\mu_r - A^\mu_s B^\nu_r - A^\nu_r B^\mu_s\right) \quad (\mu \neq \nu \text{ et } r \neq s). \tag{2.3}$$

La règle de composition (2.3) montre que toutes les matrices V de $\{V_6\}$ commutent entre elles.

Toutes les matrices V ont les mêmes vecteurs propres à six composantes que (CC); on voit facilement que les six valeurs propres de (CC) s'obtiennent en multipliant deux à deux les valeurs propres de C. Ce sont

$$a\bar{a}, \quad b\bar{b}, \quad ab, \quad \bar{a}\bar{b}, \quad a\bar{b}, \quad \bar{a}b; \tag{2.4}$$

et les vecteurs propres à six composantes se déduisent des vecteurs propres de C par une règle analogue à (2.3) par exemple :

$$M^{\mu\nu}_{ab} = M^\mu_a M^\nu_b - M^\mu_b M^\nu_a, \tag{2.5}$$

sont des composantes de $M(ab)$ le vecteur qui correspond à la valeur propre ab de (CC).

Les matrices $\overline{V}$ de l'algèbre transposée $\{\overline{V}_6\}$ ont aussi les mêmes vecteurs propres à six composantes que $\overline{(CC)}$: on voit facilement que les vecteurs $\overline{M}(ab)$ se déduisent des vecteurs propres de $\overline{C}$ par la règle savoir.

$$\overline{M}^{\mu\nu}_{ab} = \overline{M}^\mu_a \overline{M}^\nu_b - \overline{M}^\mu_b \overline{M}^\nu_a, \quad \ldots$$

Comme bases de l'algèbre $\{V_6\}$, nous pouvons prendre six matrices quelconques de V_6 linéairement indépendantes.

Nous choisissons les six matrices suivantes dont nous donnons le développement en fonction des vecteurs propres :

$$\left\{\begin{aligned}
(EE) &= V(a\bar{a} + V(b\bar{b}) + V(ab) + V(\bar{a}\bar{b}) + V(a\bar{b}) + V(\bar{a}b),\\
(CC) &= -a^2 V(a\bar{a}) - b^2 V(b\bar{b}) + ab[V(ab) + V(\bar{a}\bar{b})] - ab[V(a\bar{b}) + V(\bar{a}b)]\\
(C_2C_2) &= a^4 V(a\bar{a}) + b^4 V(b\bar{b}) + a^2b^2[V(ab) + V(\bar{a}\bar{b})] + a^2b^2[V(a\bar{b}) + V(\bar{a}b)],\\
2(EC_2) &= -2a^2 V(a\bar{a}) - 2b^2 V(b\bar{b}) + (a^2+b^2)[V(ab + V(\bar{a}\bar{b}) + V(a\bar{b}) + V(\bar{a}b)],\\
2(EC) &= -(a+b)[V(ab) - V(\bar{a}\bar{b})] + (a-b)[V(a\bar{b}) - V(\bar{a}b)],\\
2(CC_2) &= ab(a+b)[V(ab) - V(\bar{a}\bar{b})] - ab(a-b)[V(a\bar{b}) - V(\bar{a}b)],\\
& \qquad V(ab) = \overline{M}(ab)\, M(ab), \quad \ldots
\end{aligned}\right. \tag{2.6}$$

Let formules (2.6) sont obtenues d'une manière tout à fait analogue à la manière dont on a obtenu les formules (2.2).

Les quatre autres combinaisons qui peuvent être formées à partir des éléments de base de $\{C_4\}$ dépendent des six ci-dessus en vertu des quatre identités ci-dessous :

$$(2.7) \qquad \begin{cases} (EC_3)+(C\,C_2)+I_2(EC) & = 0, \\ 2(CC_3)+(C_2C_2)+I_2(CC)-I_4(EE) & = 0, \\ (C_2C_3)-I_4(EC) & = 0, \\ (C_3C_3)-I_2(C_2C_2)-I_4[(CC)+2(EC_2)] & = 0. \end{cases}$$

On montre aussi facilement qu'en multipliant V_1 et V_2 dans $\{V_6\}$ avec

$$(2.8) \qquad V_1 = (AB), \quad V_2 = (CD),$$

on a

$$V_1V_2 = \tfrac{1}{2}[(AC \cdot BD)+(AD \cdot BC)].$$

Avec (2.8), les identités (2.7) et l'identité générale de Cayley-Hamilton :

$$(2.9) \qquad C_4+I_2C_2+I_4E = 0$$

nous pouvons réduire toute matrice de forme (C_rC_s) à une fonction linéaire des six éléments de la base adoptée en (2.6).

On peut choisir des bases et démontrer des identitiés analogues pour toute algèbre $V_{\frac{n(n-1)}{2}}$ composée à partir de $\{C_n\}$ [par la manière analogue à (2.3)].

Avec les mineurs m_3 d'une matrice C_4 quelconque, nous pouvons former la matrice conjuguée D.

Dans le cas particulier de C ici étudié, où $C = Sa$ est le produit de composantes symétriques S par les antisymétriques a, on voit facilement que D est donné par

$$D = A's',$$

où A' et s' sont formés de même à partir des mineurs m_s de a et S.

Admetton que toujours

$$\| S \| = \Delta \quad \text{et} \quad S \neq 0.$$

On peut cependant avoir

$$\| C \| = 0 \quad \text{si} \quad \| a \| = 0.$$

Le cas $\| a \| = 0$ est intéressant pour la théorie du champ unitaire. Nous discuterons donc les propriétés particulières de V_6 quand $\| C \| = 0$ par suite de $\| a \| = 0$.

a est une matrice antisymétrique d'ordere 4. Donc Δ déterminant de a est un carré parfait et $\sqrt{\Delta}$ est un facteur de tous les mineurs m_3 de a_4.

Si $\| a \| = 0$, A et D mineurs conjugués de C, sont identiquement nuls.

Pour la matrice conjuguée, nous avons une identité analogue à (2.9) ;

$$(2.10) \qquad C_3+I_2C_1+D = 0 \quad (C_3 = CCC).$$

Quand

$$(2.11) \qquad \| a \| = 0, \quad I_4 = 0 \quad \text{et} \quad D = 0,$$

nous avons

$$C_3+I_2C_1 = 0$$

et des identités (2.7) nous tirons immédiatement

$$(CC_2) = 0, \quad (C_2C_2) = I_2(C_1C_1), \quad (C_3C_3) = I_2(C_2C_2). \tag{2.12}$$

Nous allons utiliser ces relations pour déduire la solution de (1.1) dans le cas particulier $I_4 = 0$.

3. Nous effectuons maintenant la sommation indiquée en (1.6) et (1.7).

Notons quelques propriétés simples du denominatéur $D\,(abc)$:

$$\begin{cases} D(a\bar{a}c) = D(ca\bar{a}) = 1-a^2, & \\ D(abc) = D(\bar{a}\bar{b}\bar{c}), & \\ D(aab) = \lambda_+-b^2, & \lambda_+ = 1+(a+b)^2, \\ D(aa\bar{b}) = \lambda_--b^2, & \lambda_- = 1+(a-b)^2, \end{cases} \tag{3.1}$$

A l'aide de ces relations simples, nous pouvons regrouper facilement les termes à sommer :

On utilise les notations ci-après pour écrire le résultat final :

$$\begin{cases} V(ab)+V(\bar{a}\bar{b}) = X, & V(a\bar{b})+V(\bar{a}b) = Y\ ; \\ V(ab)-V(\bar{a}\bar{b}) = U, & V(a\bar{b})-V(\bar{a}b) = V\ ; \\ \bar{M}_aM_a = f(a), & \bar{M}(bc)M(bc), = V(bc), \quad \ldots\ldots \end{cases} \tag{3.2}$$

La sommation (1.7) est notée

$$B = \Sigma \frac{f(a)V(bc)}{D(abc)}\,. \tag{3.3}$$

Après regroupement, il vient

$$\begin{aligned} B = \Sigma f(\lambda)\left[\frac{V(a\bar{a})}{1-a^2}+\frac{V(b\bar{b})}{1-b^2}\right]+\frac{1}{2}(X+Y)\left[\frac{f(a)+f(\bar{a})}{1-a^2}+\frac{f(b)+f(\bar{b})}{1-b^2}\right] \\ +\frac{1}{2}X\left[\frac{f(a)+f(\bar{a})}{\lambda_+-b^2}+\frac{f(b)+f(\bar{b})}{\lambda_+-a^2}\right]+\frac{1}{2}Y\left[\frac{f(a)+f(\bar{a})}{\lambda_--b^2}+\frac{f(b)+f(\bar{b})}{\lambda_--a^2}\right] \\ -\frac{1}{2}U\left[\frac{f(a)-f(\bar{a})}{1-a^2}+\frac{f(b)-f(\bar{b})}{1-b^2}\right]-\frac{1}{2}V\left[\frac{f(a)-f(\bar{a})}{1-a^2}-\frac{f(b)-f(\bar{b})}{1-b^2}\right] \end{aligned} \tag{3.4}$$

$$-\frac{1}{2}U\left[\frac{f(a)-f(\bar{a})}{\lambda_+-b^2}+\frac{f(b)-f(\bar{b})}{\lambda_+-a^2}\right]+\frac{1}{2}V\left[\frac{f(a)-f(\bar{a})}{\lambda_--b^2}-\frac{f(b)-f(\bar{b})}{\lambda_--a^2}\right].$$

Servons-nous de (2.2) et (2.6) pour exprimer X, Y, U, V et $f(a)\pm f(\bar{a})$, ...; X, Y, U, V en fonction de matrices de $\{V_6\}$ et $\{C_4\}$, et nous obtenons après un calcul simple le résultat sous sa forme définitive

$$\begin{aligned}(3.5)\qquad B = {} & E[d_0(EE)+d_1(CC)+d_2(C_2C_2)+2\mu(EC_2)]\\ & +C_2[g_0(EE)+g_1(CC)+g_2(C_2C_2)+2\nu(EC_2)]\\ & +C_1[2f_1(EC)+2h_1(CC_2)]+C_3[2f_3(EC)+2h_3(CC_2)],\end{aligned}$$

où les coefficients d, g, μ, ν, f, h sont fonctions des seuls invariants de C. Si

$$t_0 \equiv 1+I_2+I_4,\quad t_+ \equiv \lambda_+^2+I_2\lambda_++I_4,\quad t_- = \lambda_-^2+I_2\lambda_-+I_4;$$

$$\frac{t_+-t_-}{2} \equiv x = 1+5I_4-I_2,\ \frac{t_+-t_-}{4ab} \equiv y = 2-I_2$$

et

$$t_+t_- = (1-I_2+5I_4)^2-4I_4(2-I_2)^2$$

sont fonctions des invariants.

Les coefficients de (3.5) sont

$$(3.6)\left\{\begin{aligned} & d_0 = \frac{1+I_2}{t_0}-\frac{I_4y}{t_+t_-}-\frac{I_4(x-2y)}{t_0t_+t_-},\\ & d_1 = \frac{x-y(y-1)}{t_+t_-},\ d_2 = -\frac{y}{t_+t_-}+\frac{2y-x}{t_0t_+t_-};\\ & g_0 = \frac{1_4(x-2y)}{t_0t_+t_-},\ g_1 = \frac{y}{t_+t_-},\ g_2 = \frac{x-2y}{t_0t_+t_-},\ 2f_3 = \frac{2[2x+y(2I_2-t_0)]}{t_0t_+t_-};\\ & 2h_3 = -\frac{4(x+yI_2)}{t_0t_+t_-},\ 2\mu = \frac{I}{t_0}+\frac{x}{t_+t_-}-\frac{2(x-2yI_4)}{t_0t_+t_-},\ 2\nu = \frac{2(x-2I_4y)}{t_0t_+t_-};\\ & 2f_1 = \frac{2[f_3I_2+41_4y-x(1+I_4)]}{t_0t_+t_-}\quad \text{et}\quad 2h_1 = -\frac{4[I_2h_3+x-y(1+I_4)]}{t_0t_+t_-}.\end{aligned}\right.$$

4. Nous avons exprimé le résultat final (3.6) en fonction de matrices déduites de C suivant des lois déterminées, avec des coefficients qui sont fonctions des invariants de C. Ce résultat est valuable dans tous les cas, même quand pour des matrices parti-

culieres, il n'existe pas un système complet de vecteurs propres et l'on ne peut plus développer suivant (3.3) la matrice résolvante B mais le résultat général (3.5) reste toujours vrai. L'expression (3.6) donne donc la solution générale du problème posé en (1.1).

Néanmoins, il est intéressant d'observer que, dans les cas particuliers $I_4 = 0$ ou $I_2 = 0$, $I_4 = 0$, on peut facilement mettre la solution sous forme d'une somme d'un petit nombre de termes, grâce à une méthode d'itération.

Discutons brièvement le cas $I_4 = 0$.

On a déjà déduit les propriétés spéciales de la matrice C en (2.11) et (2.12) : à savoir

$$(CC_2) = 0, \quad (C_2C_2) = I_2(C_1C_1), \quad (C_3C_3) = I_2(C_2C_2),$$
$$C_3+\mathrm{I}_2C_1 = 0, \qquad C_4+I_2C_2 = 0.$$

Nous écrivons (1.1) dans la forme

$$U = AT = [E\{(EE)+(CC)\}+2C(EC)]T$$

ou

(4.1). $$U = T+[E(CC)+2C(EC)]\,T = T+\delta T;$$

et cherchons une solution en posant

$$T = U+\Phi_1.$$

L'équation ayant la forme

$$U \equiv T+\delta T.$$

On déduit facilement pour Φ, l'equation

$$-\delta U = \Phi_1+\delta\Phi_1.$$

En répétant l'opération, nous obtenons successivement

(4.2) $$\delta\delta U = \Phi_2+\delta\Phi_1, \quad -\delta\delta\delta U = \Phi_3+\delta\Phi_3.$$

En raison des propriétés spéciales de l'opérateur [(2.11)—(2.12)] nous déduisons facilment

(4.3) $$\delta\delta\delta = I_2^2\delta.$$

En posant $\delta = \Delta-E$, où E est l'opérateur-unité, l'équation (4.2) s'écrit

$$\Delta\Phi_3 = -I_2^2(\Delta-E)U$$

ou

(4.4) $$\Delta(\Phi_3+I_2^2U) = I_2^2U.$$

L'équation originale est

$$\Delta T = U.$$

Comme dét $\Delta \neq 0$, la solution est unique et l'on voit facilement que la solution de (4.4) est

$$\Phi_3+I_2^2U = I_2^2T.$$

En écrivant

$$\Phi_3 = T-U+\delta U-\delta\delta U = I_2^2(T-U),$$

nous avons facilement

$$T = U+\frac{\delta U-\delta\delta U}{I_2^2-1}\ . \tag{4.5}$$

En développant, nous avons le résultat suivant :

$$T_{\sigma\mu\nu} = U_{\sigma\mu\nu}-(1+I_2)^{-1}U_{\sigma kl}C_\mu^k\ C_\nu^l+(I_2^2-1)^{-1}[C_\sigma^t(C_\mu^kU_{tk\nu}+C_\nu^lU_{t\mu l})]$$

$$-(I_2^2-1)^{-1}C_{2,\sigma}^t\ (U_{tk\nu}C_{2,\mu}^k+U_{t\mu l}C_{2,\nu}^l)-2(I_2^2-1)^{-1}C_{2,\sigma}^tC_\mu^kC_\nu^lU_{tkl}\,.$$

La solution générale nous donne le même résultat en posant $I_4 = 0$ et en utilisant les relations particulières (2.11) et (2.12).

On peut aussi traiter de même les autres cas $I_2 = 0$, $I_4 = 0$.

POST-SCRIPTUM.

Si les coefficients λ de

$$N \equiv \lambda_0+\lambda_1C_1+\lambda_2C_2+\lambda_3C_3 \text{ (nombre quelconque dans } [C]_4)$$

sont choisis dans l'algèbre $[V]_6$, ces nombres forment une nouvele algèbre $[cV]_{24}$.

Résoudre

$$U = [e(EE)+e(cc)+2c(Ec)]\ T \equiv BT$$

est donc équivalent à trouber l'inverse de B en tant que nombre algébrique. L'inverse est aussi un nomber dans $[cV]$ et on peut le trouver facilement de la facon suivante.

Nous vérifions d'abord les résultats suivants à partir de la règle de multiplication et des identités pour l'algèbre $[V]_6$.

Si

$$L = (C_2C_2)-I_4(EE), \qquad M = 2(EC_2)+I_2(CC),$$

$$R = 2L+I_2M, \qquad X = 4(EC)^2 = 2(CC)+2(EC_2),$$

et $D \equiv I_2^2-4I_4$, le discriminant des équations

$$x^2+I_2x+I_4 = 0,$$

on a alors

$$L(EC) = M(EC) = 0; \quad d'où\ LX = MX = RX = 0$$

$$RL = DL, \quad RM = DM, \quad RR = DR, \quad M^2 = R$$

et

$$XX+2I_2X+D = R.$$

Dans ce qui suit, nous employons les abréviations suivantes : $s = 2+I_2$, $t_0 = 1+I_2+I_4$, $x = 1+5I_4-I_2$, et $y = 2-I_2$.

Comme $B = (EE)+(CC)+2C(EC)$, en le multipliant par son conjugé

$$B^* = (EE)+(CC)-2C(EC),$$

nous avons

$$BB^* = (EE+CC)^2-4C_2(EC)^2$$

ou

$$BB^* = \left(1+ \frac{R-sM}{t_0}\right)(t_0+X-C_2X)$$

après factorisation;

Comme
$$\left(1+\frac{R-sM}{t_0}\right)\left(1+\frac{R+sM}{t_0}\right) = 1,$$

$$\frac{1}{BB^*} = \frac{\left[1+\frac{R+sM}{2t_0}\right][t_0+X(1+I_2)+C_2X]}{[t_0+X-C_2X][t_0+X(1+I_2)+C_2X]}$$

après l'addition du même facteur dans le numérateur et le dénominateur de l'expression de droite.

En faisant la multiplication, on obtient pour le dénominateur

$$t_0[x+y(X+I_2)+R].$$

Nous observons que

$$[x+y(X+I_2)+R][x-y(X+I_2)-\nu R]$$
$$= \Omega = t_+t_- = x^2-4I_4y^2$$

si
$$\nu = \frac{x-2y}{t_0}.$$

On peut donc exprimer facilement l'inverse par l'équation suivante :

$$B^{-1} = \frac{B^*\left[1+\frac{R+sM}{2t_0}\right][t_0+X(1+I_2)+C_2X][x-y(X+I_2)-\nu R]}{t_0t_+t_-}.$$

En supprimant les crochets, on obtient un résultat équivalent à celui de l'article principal.

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Solution of a Tensor Equation Occurring in the Unitary Field Theory

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Summary — The solution to the tensor equation on which the determination of $\Gamma^{\lambda}_{\mu\nu}$ depends, is explicitly expressed in terms of the C and their invariants.

Note that the iteration gives a rigorous solution when

$$\det C = 0.$$

1. In a preceding article (¹) it has been shown that it is possible to reduce the calculation of the affine coefficients $\Gamma^{\lambda}_{\mu\nu}$ in terms of $g^{\mu\nu}$ and their derivatives to the solution of the tensor equation

$$U_{a\mu\nu} = T_{a\mu\nu} + T_{akl} C^k_\mu C^l_\nu + C^t_a (T_{tk\nu} C^k_\mu + T_{t\mu l} C^l_\nu). \tag{1.1}$$

The tensors U and T are antisymmetric with regard to the indices μ and ν, C^k_μ are the elements of a matrix C formed from the symmetric and antisymmetric parts of the tensor g :

$$\left\{ \begin{aligned} & C = Sa, \quad a_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - g_{\nu\mu}), \quad s_{tk} = \frac{1}{2}(g_{tk} + g_{kt}), \\ & \qquad\qquad S^{\mu t} S_{tk} = \delta^\mu_k. \end{aligned} \right. \tag{1.2}$$

The eigenvalues x of the matrix C satisfy the equation

$$x^4 + I_2 x^2 + I_4 = 0, \tag{1.3}$$

I_2 and I_4 are even invariants of C ; from the equation (1.2) it immediately follows that the odd invariants are zero.

The eigenvector M^k_a of C corresponding to the eigenvalue a satisfies the equation

$$C^\mu_k M^k_a = a M^\mu_a. \tag{1.4}$$

The transposed matrix $\overline{C}$ ($\overline{C}^\mu_k = C^k_\mu$) has the same eigenvalues as C but its eigenvectors $\overline{M}^k_a$ are different.

If the characteristic equation has all its roots distinct, then there exists a complete series of eigenvectors M and $\overline{M}$ of C and $\overline{C}$ and we can express the elements of C in terms of the eigenvectors : that is

$$C^k_\mu = \Sigma\, \lambda \overline{M}^\mu_\lambda M^k_\lambda. \tag{1.5}$$

The solution is easily calculated from (1.1) :

$$(1.6)\qquad \begin{cases} T_{\sigma\mu\nu} = \sum\limits_{abc} (1+ab+bc+ca)^{-1}\overline{M}{}^{\sigma}_{a}M^{x}_{a}\overline{M}\begin{pmatrix}\mu\nu\\ bc\end{pmatrix}M\begin{pmatrix}yz\\ bc\end{pmatrix}U_{xyz}\ \ (b\neq c), \\ \\ M\begin{pmatrix}yz\\ bc\end{pmatrix} = M^{y}_{b}M^{z}_{c} - M^{y}_{c}M^{z}_{b}\ ; \end{cases}$$

and the summation is over the 24 possible combinations of (abc).

The solution (1.6) can be written in the form of a tensor equation

the components of the tensor B being

$$(1.7)\qquad B^{xyz}_{\sigma\mu\nu} = \Sigma\ \frac{\overline{M}{}^{\sigma}_{a}\overline{M}\begin{pmatrix}\mu\nu\\ bc\end{pmatrix}M^{x}_{a}M\begin{pmatrix}yz\\ bc\end{pmatrix}}{1+bc+ca+ab}$$

(1.7) is identical to (1.5).

I intend to carry out the summation in (1.7) and obtain the components of B in terms of the elements of C.

2. By means of C and of the unitary matrix E we can form an algebra $\{C_4\}$ with addition and matrix multiplication.

The general formula of a matrix A in $\{C_4\}$ is

$$(2.1)\qquad \begin{cases} A = \lambda_0+\mu_1C_1+\mu_2C_2+\mu_3C_3; \\ C_1 = C,\quad C_2 = CC,\quad C_3 = CCC. \end{cases}$$

All the matrices A commute and therefore have the same eigenvector M as C.

The matrices $\overline{\mathrm{A}}$ of the transposed algebra also have the same eigenvector $\overline{\mathrm{M}}$ as $\overline{\mathrm{C}}$.

If a , $\bar{a}$, b , $\bar{b}$ are the roots of the characteristic equation (1.3), one can easily see that the basic matrices develop in the following way :

$$(2.2)\qquad \begin{cases} E = f(a)+f(\bar{a})+f(b)+f(\bar{b}); & C_2 = a^2[f(a)+f(\bar{a})]+b^2[f(b)+f(\bar{b})]; \\ C_1 = a[f(a)-f(\bar{a})+b[f(b)+f(\bar{b})]; & C_3 = a^3[(fa)-f(\bar{a})]+b^3[f(b)-f(\bar{b})]; \end{cases}$$

where $f\ (a) = \overline{\mathrm{M}}_a\ \mathrm{M}_a,. .$ and by adding the lower and upper indices to the above formulae, one gets

$$C^{\lambda}_{\mu} = \left(\overline{M}{}^{\mu}_{a}M^{\lambda}_{a} - \overline{M}{}^{\mu}_{\bar{a}}M^{\lambda}_{\bar{a}}\right)a + b\left(\overline{M}{}^{\mu}_{b}M^{\lambda}_{b} - \overline{M}{}^{\mu}_{\bar{b}}M^{\lambda}_{\bar{b}}\right)$$

Let us now deduce a new algebra $\{V_6\}$, where the matrix V is formed from any two elements A, B of $\{C_4\}$:

$$V = (AB),$$

with

(2.3)

$$V^{\mu\nu}_{rs} = \frac{1}{2}\ (A^{\mu}_{r}\ B^{\nu}_{s} + A^{\nu}_{s}\ B^{\mu}_{r} - A^{\mu}_{s}\ B^{\nu}_{r} - A^{\nu}_{r}\ B^{\mu}_{s})\quad (\mu\neq\nu \text{ et } r\neq s).$$

The rule of composition (2.3) shows that all the matrices V of $\{V_6\}$ commute among themselves.

All the matrices V have the same eigenvectors with six components as (CC) ; it is easily seen that the six eigenvalues of (CC) are obtained by multiplying the eigenvalues of C two by two. These are

$$a\bar{a}, \quad b\bar{b}, \quad ab, \quad \bar{a}\bar{b}, \quad a\bar{b}, \quad \bar{a}b; \tag{2.4}$$

and the eigenvectors with six components are deduced from the eigenvectors of C through a similar rule to (2.3) ; for example :

$$M^{\mu\nu}_{ab} = M^{\mu}_{a}\, M^{\nu}_{b} - M^{\mu}_{b} M^{\nu}_{a}, \tag{2.5}$$

are the components of Mab, the vector which corresponds to the eigenvalue ab of (CC).

The matrices $\overline{V}$ of the transposed algebra $\{\overline{V}_6\}$, also have the same eigenvectors with six components as ($\overline{C}\,\overline{C}$) ; it is easily noticed that the vectors $\overline{M}$ (ab) are deduced from the eigenvectors of $\overline{C}$ by the following rule :

$$\overline{M}^{\mu\nu}_{ab} = \overline{M}^{\mu}_{a}\, \overline{M}^{\nu}_{b} - \overline{M}^{\mu}_{b}\, \overline{M}^{\nu}_{a}, \quad \ldots$$

As the bases of the algebra $\{V_6\}$, we can take any six linearly independent matrices of V_6.

We choose the following six matrices whose expansion we give in terms of the eigenvectors :

$$(2.6) \quad \left\{ \begin{aligned} (EE) &= V(a\bar{a} + V(b\bar{b}) + V(ab) + V(\bar{a}\bar{b}) + V(ab) + V(\bar{a}b), \\ (CC) &= -a^2 V(a\bar{a}) - b^2 V(b\bar{b}) + ab[V(ab) + V(\bar{a}\bar{b})] - ab[V(ab) + V(\bar{a}b)] \\ (C_2C_2) &= a^1 V(a\bar{a}) + b^4 V(b\bar{b}) + a^2b^2[V(ab) + V(\bar{a}\bar{b})] + a^2b^2[V(a\bar{b}) + V(\bar{a}b)], \\ 2(EC_2) &= -2a^2 V(a\bar{a}) - 2b^2 V(b\bar{b}) + (a^2+b^2)[V(ab + V(\bar{a}\bar{b}) + V(a\bar{b}) + V(\bar{a}b)], \\ 2(EC) &= -(a+b)[V(ab) - V(\bar{a}\bar{b})] + (a-b)[V(a\bar{b}) - V(\bar{a}b)], \\ 2(CC_2) &= ab(a+b)[V(ab) - V(\bar{a}\bar{b})] - ab(a-b)[V(a\bar{b}) - V(\bar{a}b)], \\ & \qquad V(ab) = \overline{M}(ab)\, M(ab), \quad \ldots \end{aligned} \right.$$

The formulae (2.6) are obtained in a manner completely similar to the manner used to obtain the formulas (2.2).

The four other combinations which can be formed from the basic elements of $\{C_4\}$ depend on the six above mentioned equations by virtue of the four identities mentioned below :

$$(2.7) \quad \left\{ \begin{aligned} &(EC_3) + (C\,C_2) + I_2(EC) &&= 0, \\ &2(CC_3) + (C_2C_2) + I_2(CC) - I_1(EE) &&= 0, \\ &(C_2C_3) - I_4(EC) &&= 0, \\ &(C_3C_3) - I_2(C_2C_2) - I_4[(CC) + 2(EC_2)] &&= 0. \end{aligned} \right.$$

It is also easily shown that by multiplying V_1 and V_2 in $\{V_6\}$ with

we have
$$V_1 = (AB), \quad V_2 = (CD),$$

(2.8)
$$V_1V_2 = \tfrac{1}{2}[(AC \cdot BD)+(AD \cdot BC)].$$

With (2.8), the identities in (2.7) and the general identity of Cayley - Hamilton :

(2.9)
$$C_4+I_2C_2+I_4E = 0$$

each matrix of the form ($C_r\ C_s$) can be reduced to a linear function of the six elements of the basis adopted in (2.6).

We can choose the bases and demonstrate the analogous identities for all algebras $V_{\frac{n(n-1)}{2}}$ composed from $\{C_n\}$ [by a process similar to (2.3)].

With the minors m_3 of any matrix C_4, a conjugate matrix D can be formed.

In the case of C studied here, where $C = Sa$ is the product of the symmetric component S by the antisymmetric a, it is easily seen that D is given by
$$D = A's',$$
where A′ and s' are formed similarly from the minors m_3 of a and S.

We take for granted that always
$$\| S \| = \Delta \quad \text{et} \quad S \neq 0.$$

One can nevertheless have
$$\| C \| = 0 \quad \text{si} \quad \| a \| = 0.$$

The case $\| a \| = 0$ is interesting for the unitary field theory. Therefore we will discuss the particular properties of V_6 when $\| C \| = 0$ as a consequence of $\| a \| = 0$.

a is an antisymmetric matrix of order 4. Therefore Δ which is the determinant of a is a perfect sequare and $\sqrt{\Delta}$ is a factor of all the minors m_3 of a.

If $\| a \| = 0$, A and D, conjugate minors of C, are identically zero. For the conjugate matrix, we have an identity analogous to (2.9) ;

(2.10)
$$C_3+I_2C_1+D = 0 \quad (C_3 = CCC).$$

When
$$\| a \| = 0, \quad I_4 = 0 \quad \text{et} \quad D = 0,$$
we have

(2.11)
$$C_3+I_2C_1 = 0$$

and from the identities (2.7) we immediately obtain

(2.12)
$$(CC_2) = 0, \quad (C_2C_2) = I_2(C_1C_1), \quad (C_3C_3) = I_2(C_2C_2).$$

We will use these relations to deduce the solution of (1.1) in the particular case $I_4 = 0$.

3. Now we carry out the summation indicated in (1.6) and (1.7). Let us note some simple properties of the denominator D (*abc*) :

$$(3.1)\qquad \begin{cases} D(a\bar{a}c) = D(ca\bar{a}) = 1-a^2, \\ D(abc) = D(\bar{a}\bar{b}\bar{c}), \\ D(aab) = \lambda_+ - b^2, \qquad \lambda_+ = 1+(a+b)^2, \\ D(aa\bar{b}) = \lambda_- - b^2, \qquad \lambda_- = 1+(a-b)^2, \end{cases}$$

With the help of these simple relations, we can easily regroup the terms to be summed up :

We use the following notations to write the final results :

$$(3.2)\qquad \begin{cases} V(ab)+V(\bar{a}\bar{b}) = X, \qquad V(a\bar{b})+V(\bar{a}b) = Y\ ; \\ V(ab)-V(\bar{a}\bar{b}) = U, \qquad V(a\bar{b})-V(\bar{a}b) = V\ ; \\ \bar{M}_a M_a = f(a), \qquad \bar{M}(bc)M(bc), = V(bc), \quad \ldots\ldots \end{cases}$$

The summation (1.7) is written as

$$(3.3)\qquad B = \Sigma \frac{f(a)V(bc)}{D(abc)}\,.$$

After regrouping we get

$$(3.4)\qquad B = \Sigma f(\lambda)\left[\frac{V(a\bar{a})}{1-a^2}+\frac{V(b\bar{b})}{1-b^2}\right]+\frac{1}{2}(X+Y)\left[\frac{f(a)+f(\bar{a})}{1-a^2}+\frac{f(b)+f(b)}{1-b^2}\right]$$

$$+\frac{1}{2}X\left[\frac{f(a)+f(\bar{a})}{\lambda_+-b^2}+\frac{f(b)+f(\bar{b})}{\lambda_+-a^2}\right]+\frac{1}{2}Y\left[\frac{f(a)+f(\bar{a})}{\lambda_--b^2}+\frac{f(b)+f(\bar{b})}{\lambda_--a^2}\right]$$

$$-\frac{1}{2}U\left[\frac{f(a)-f(\bar{a})}{1-a^2}+\frac{f(b)-f(\bar{b})}{1-b^2}\right]-\frac{1}{2}V\left[\frac{f(a)-f(\bar{a})}{1-a^2}-\frac{f(b)-f(\bar{b})}{1-b^2}\right]$$

$$-\frac{1}{2}U\left[\frac{f(a)-f(\bar{a})}{\lambda_+-b^2}+\frac{f(b)-f(\bar{b})}{\lambda_+-a^2}\right]+\frac{1}{2}V\left[\frac{f(a)-f(\bar{a})}{\lambda_--b^2}-\frac{f(b)-f(\bar{b})}{\lambda_--a^2}\right].$$

Let us make use of (2.2) and (2.6) to express X, Y, U, V and $f(a) \pm f(\bar{a}), \ldots$; X, Y, U,V are functions of matrices of {V6} and $\{C_4\}$, and we obtainn after an easy calculation the result in its final form

$$
\begin{aligned}
B = {} & E[d_0(EE)+d_1(CC)+d_3(C_2C_2)+2\mu(EC_2)] \\
& +C_2[g_0(EE)+g_1(CC)+g_2(C_2C_2)+2\nu(EC_2)] \\
& +C_1[2f_1(EC)+2h_1(CC_2)]+C_3[2f_3(EC)+2h_3(CC_2)],
\end{aligned}
\tag{3.5}
$$

where the coefficient *d, g,* μ, ν, *f, h* are functions only of the invariants of C. If

$$t_0 \equiv 1+I_2+I_4, \quad t_+ \equiv \lambda_+^2+I_2\lambda_++I_4, \quad t_- = \lambda_-^2+I_2\lambda_-+I_4;$$

$$\frac{t_+-t_-}{2} \equiv x = 1+5I_4-I_2, \ \frac{t_+-t_-}{4ab} \equiv y = 2-I_2$$

and

$$t_+t_- = (1-I_2+5I_4)^2-4I_4(2-I_2)^2$$

are functions of the invariants

The coefficients of (3.5) are

$$
\left\{
\begin{aligned}
& d_0 = \frac{1+I_2}{t_0} - \frac{I_4y}{t_+t_-} - \frac{I_4(x-2y)}{t_0t_+t_-}, \\
& d_1 = \frac{x-y(y-1)}{t_+t_-}, \ d_2 = -\frac{y}{t_+t_-} + \frac{2y-x}{t_0t_+t_-}; \\
& g_0 = \frac{1_4(x-2y)}{t_0t_+t_-}, \ g_1 = \frac{y}{t_+t_-}, \ g_2 = \frac{x-2y}{t_0t_+t_-}, \ 2f_3 = \frac{2[2x+y(2I_2-t_0)]}{t_0t_+t_-}; \\
& 2h_3 = -\frac{4(x+yI_2)}{t_0t_+t_-}, \ 2\mu = \frac{I}{t_0} + \frac{x}{t_+t_-} - \frac{2(x-2yI_4)}{t_0t_+t_-}, \ 2\nu = \frac{2(x-2I_4y)}{t_0t_+t_-}; \\
& 2f_1 = \frac{2[f_3I_2+41_4y-x(1+I_4)]}{t_0t_+t_-} \quad \text{et} \quad 2h_1 = -\frac{4[I_2h_3+x-y(1+I_4)]}{t_0t_+t_-}.
\end{aligned}
\right.
\tag{3.6}
$$

4. We have expressed the final result (3.6) in terms of the matrices deduced from C following fixed rules, with coefficients which are functions of the invariants of C. This result is valid in all the cases, even when for particular matrices there is no complete system of eigenvectors and we can no longer develop the resolvent matrix B as per (3.3) but the general result (3.5) always remains true. The expression (3.6) thus gives the general solution of the problem posed in (1.1).

Nevertheless it is interesting to observe that in the particular cases $I_4 = 0$ or $I_2 = 0$, $I_4 = 0$, we can easily put the solution in the form of a total sum of a small number of terms, thanks to a method of iteration.

Let us briefly discuss the case $I_4 = 0$.

We have already deduced the special properties of the matrix C in (2.11) and (2.12) : to wit

$$(CC_2) = 0, \quad (C_2C_2) = I_2(C_1C_1), \quad (C_3C_3) = I_2(C_2C_2),$$
$$C_3+\mathrm{I}_2C_1 = 0, \qquad C_4+I_2C_2 = 0.$$

We write (1.1) in the form

$$U = AT = [E\{(EE) + (CC)\}+2C(EC)]T$$

or

(4.1) $$U = T+[E(CC)+2C(EC)]\,T = T+\delta T;$$

and look for a solution by putting

$$T = U+\Phi_1.$$

The equation having the form

$$U \equiv T+\delta T.$$

we easily deduce for Φ the equation

$$-\delta U = \Phi_1+\delta\Phi_1.$$

By repeating the operation, we successively obtain

(4.2) $$\delta\delta U = \Phi_2+\delta\Phi_1, \quad -\delta\delta\delta U = \Phi_3+\delta\Phi_3.$$

Beacause of special properties of the operator [(2.11) - (2.12)] we easily deduce

(4.3) $$\delta\delta\delta = I_2^2\delta.$$

By putting $\delta = \Delta - E$, where E is the unit operator, the equation (4.2) is written as

$$\Delta\Phi_3 = -I_2^2(\Delta-E)U$$

or

(4.4) $$\Delta(\Phi_3+I_2^2U) = I_2^2U.$$

The original equation is

$$\Delta T = U.$$

With det $\Delta \neq 0$, the solution is unique and one easily sees that the solution of (4.4) is

$$\Phi_3+I_2^2U = I_2^2T.$$

By writing

$$\Phi_3 = T-U+\delta U-\delta\delta U = I_2^2(T-U),.$$

we easily have

(4.5) $$T = U+ \frac{\delta U-\delta\delta U}{I_2^2-1}\ .$$

By developing this, we have the following result :

$$T_{\sigma\mu\nu} = U_{\sigma\mu\nu}-(1+I_2)^{-1}U_{\sigma kl}C^k_\mu\, C^l_\nu+(I_2^2-1)^{-1}[C^t_\sigma(C^k_\mu U_{tk\nu}+C^l_\nu U_{t\mu l})]$$

$$-(I_2^2-1)^{-1}C^t_{2,\sigma}\,(U_{tk\nu}C^k_{2,\mu}+U_{t\mu l}C^l_{2,\nu})-2(I_2^2-1)^{-1}C^t_{2,\sigma}C^k_\mu C^l_\nu U_{tkl}\,. \tag{4.6}$$

The general solution gives us the same result by putting $I_4 = 0$, and using the particular relations (2.11) and (2.12).

We can similarly treat the other cases $I_2 = 0$, $I_4 = 0$.

Post-script

If the coefficients λ of

$$N \equiv \lambda_0+\lambda_1C_1+\lambda_2C_2+\lambda_3C_3 \quad \text{(any number in } [C]_4)$$

are chosen in the algebra $[V]_6$, these numbers form a new algebra $[cV]_{24}$.

To solve

$$U = [e(EE)+e(cc)+2c(Ec)]\,T \equiv BT$$

is thus equivalent to finding the inverse of B as an algebraic number. The inverse is also a number in [cV] and we can easily find it in the following way.

First, we verify the following results as per the rule of multiplication and the identities for the algebra $[V]_6$.

If

$$L = (C_2C_2)-I_4(EE), \qquad M = 2(EC_2)+I_2(CC),$$

$$R = 2L+I_2M, \qquad X = 4(EC)^2 = 2(CC)+2(EC_2),$$

and $D \equiv I_2^2 - 4I_4$ is the discriminant of the equations

$$x^2+I_2x+I_4 = 0,$$

then we have

$$L(EC) = M(EC) = 0; \quad d\text{'où}\, LX = MX = RX = 0$$

$$RL = DL, \quad RM = DM, \quad RR = DR, \quad M^2 = R$$

and

$$XX+2I_2X+D = R.$$

In what follows we use the the following abbreviations : $s = 2 + I_2$, $t_o = 1 + I_2 + I_4$ $x = 1 + 5I_4 - I_2$, and $y = 2 - I_2$.

With B = (EE) + (CC) + 2C(EC), by multiplying it with its conjugate

$$B^* = (EE)+(CC)-2C(EC),$$

we have

$$BB^* = (EE+CC)^2-4C_2(EC)^2$$

or

$$BB^* = \left(1+\frac{R-sM}{t_0}\right)(t_0+X-C_2X)$$

after factorization ; with $\left(1-\frac{\mathrm{R}-s\mathrm{M}}{t_o}\right)\left(1+\frac{\mathrm{R}+s\mathrm{M}}{t_o}\right) \equiv t$,

$$\frac{1}{BB^*}=\frac{\left[1+\frac{R+sM}{2t_0}\right][t_0+X(1+I_2)+C_2X]}{[t_0+X-C_2X][t_0+X(1+I_2)+C_2X]}$$

after adding the same factor in the numerator and denominator of the right side of the equation.

By multiplying we obtain for the denominator

$$t_0[x+y(X+I_2)+R].$$

We notice that

$$[x+y(X+I_2)+R][x-y(X+I_2)-\nu R]$$
$$=\Omega=t_+t_-=x^2-4I_4y^2$$

if $\nu=\frac{x-2y}{t_o}$.

We can thus easily express the inverse by the following equation :

$$B^{-1}=\frac{B^*\left[1+\frac{R+sM}{2t_0}\right][t_0+X(1+I_2)+C_2X][x-y(X+I_2)-\nu R]}{t_0t_+t_-}.$$

By eliminating the brackets, we obtain a result equivalent to the one in the main paper.

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